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Twistors and black holes

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Abstract: Motivated by black hole physics in $\mathcal{N} = 2, D = 4$ supergravity, we study the geometry of quaternionic-Kähler manifolds $\mathcal{M}$ obtained by the c-map construction from projective special Kähler manifolds $\mathcal{M}_s$. Improving on earlier treatments, we compute the Kähler potentials on the twistor space $\mathcal{Z}$ and Swann space $\mathcal{S}$ in the complex coordinates adapted to the Heisenberg symmetries. The results bear a simple relation to the Hesse potential Σ of the special Kähler manifold $\mathcal{M}_s$, and hence to the Bekenstein-Hawking entropy for BPS black holes. We explicitly construct the "covariant c-map" and the "twistor map", which relate real coordinates on $\mathcal{M} \times \mathbb{CP}^1$ (resp. $\mathcal{M} \times \mathbb{R}^4/\mathbb{Z}_2$) to complex coordinates on $\mathcal{Z}$ (resp. $\mathcal{S}$). As applications, we solve for the general BPS geodesic motion on $\mathcal{M}$, and provide explicit integral formulae for the quaternionic Penrose transform relating elements of $H^1(\mathcal{Z}, \mathcal{O}(-k))$ to massless fields on $\mathcal{M}$ annihilated by first or second order differential operators. Finally, we compute the exact radial wave function (in the supergravity approximation) for BPS black holes with fixed electric and magnetic charges.

Keywords: Supergravity Models, Black Holes in String Theory.

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Contents

1. Introduction

In this paper, we study the geometry of quaternionic-Kähler manifolds $\mathcal{M}$ obtained by the c-map construction of [1, 2] from a projective special Kähler manifold $\mathcal{M}_s$. While our results may be of independent mathematical interest, our motivation stems from the physics of BPS black holes in $\mathcal{N} = 2, D = 4$ supergravity, as we now explain. The mathematically oriented or impatient reader is kindly urged to proceed to section 1.2.

1.1 Motivation

Supersymmetric black holes in Type II string theory compactified on a Calabi-Yau three-fold X offer a rich playground to test the stringy description of black-hole micro-states beyond leading order: on the macroscopic side, thanks to an off-shell superspace description of vector multiplets, an infinite series of higher-derivative curvature corrections can be computed using the topological string on X [3, 4]; on the microscopic side, the weakly coupled D-brane [5, 6] or M5-brane [7] description can be extended to strong coupling, thanks to the tree-level decoupling between vector multiplets and hypermultiplets. The interplay between these two descriptions has culminated in a recent conjecture [8] relating the microscopic degeneracies, to all orders in an expansion in the inverse of the graviphoton charge, to the topological string amplitude (see e.g. [9] for a recent review and further references).

Due to the aforementioned decoupling between vector multiplets and hypermultiplets, the study of BPS black holes in $\mathcal{N} = 2$ supergravity is usually framed in the language of special geometry. It has however become increasingly clear that quaternionic-Kähler geometry may be a more useful framework. Indeed, the attractor equations that govern the radial evolution of the complex vector multiplet scalars in the black hole geometry are equivalent to "supersymmetric" geodesic motion on a para-quaternionic-Kähler manifold $\mathcal{M}^*$, of dimension $4n$ (where $n - 1$ is the number of vector multiplets) and split signature [10]. This $\mathcal{M}^*$ is a particular analytic continuation (studied in [10, 11]) of the (positive signature) quaternionic-Kähler manifold $\mathcal{M}$, obtained via the c-map construction of [1, 2] from the projective special Kähler manifold $\mathcal{M}_s$ describing the vector multiplet scalars in four dimensions. This description of the attractor equations follows from the fact that stationary black holes in four dimensions can be reduced to three dimensions along their timelike isometry, where they become solutions of three-dimensional Euclidean gravity coupled to a non-linear sigma model on $\mathcal{M}^*$. Further assuming spherical symmetry leads to geodesic motion on $\mathcal{M}^*$ [12]; the electric, magnetic and NUT charges of the black hole are identified as conserved Noether charges for a Heisenberg algebra of isometries of $\mathcal{M}^*$. This equivalence between black hole attractor equations and geodesic motion on a quaternionic-Kähler manifold can also be seen as a consequence of T-duality along the time direction, which relates black holes to D-instantons, with a non-trivial radial dependence of the hypermultiplets in the dual four-dimensional theory [13 – 15].

This reformulation is particularly well suited to the radial quantization of BPS black holes, which, according to the proposal in [16], should provide a holographic point of view on the conjecture of [8]. Indeed, once the radial evolution equations are reformulated as geodesic motion, quantization could proceed as usual by replacing functions on the classical phase space $T^*(\mathcal{M}^*)$, of real dimension $8n$, by (square integrable) wave functions on $\mathcal{M}$, satisfying the appropriate Wheeler-De Witt type constraint [10]. The corresponding Hilbert space is infinite dimensional, even after restricting to the subspace with fixed electric and magnetic charges.

More relevant however is the quantization of supersymmetric geodesic motion, corresponding to BPS black holes: the analysis in [17] (as announced in [10, 9]) shows that after imposing the BPS constraints the classical phase space becomes the twistor space $\mathcal{Z}$ of $\mathcal{M}$, with real dimension $4n+2$, almost twice as small as the non-BPS phase space. The twistor space is a standard construct in quaternionic geometry [18], which carries a Kähler-Einstein metric and a canonical integrable complex structure, unlike the base $\mathcal{M}$ whose quaternionic structure has a non-vanishing Nijenhuis tensor. It is fibered by 2-spheres over $\mathcal{M}$; physically, the coordinate in the fiber keeps track of the projectivized Killing spinor preserved by the black hole [17]. In fact, it is convenient to integrate the entire quaternionic structure on $\mathcal{M}$ by introducing an $\mathbb{R}^4/\mathbb{Z}_2$ bundle $\mathcal{S}$ over the quaternionic-Kähler space $\mathcal{M}$, known as the the "hyperkähler cone" or "Swann space" [19]; the twistor space $\mathcal{Z}$ then arises as a Kähler quotient $\mathcal{S}//\,\mathrm{U}(1)$. This construction is particularly natural in the conformal approach to supergravity [20], and leads to a simple description of the supersymmetric geodesics in terms of holomorphic maps from $\mathbb{C}$ to $\mathcal{S}$ [17].

Having identified the twistor space $\mathcal{Z}$ as the BPS phase space, quantization proceeds in the usual way for Kähler manifolds, i.e. by replacing functions on $\mathcal{Z}$ with classes in the cohomology of an appropriate line bundle over $\mathcal{Z}$. In more mundane terms, the BPS Hilbert space consists of holomorphic functions in $2n+1$ variables. In stark contrast to the non-BPS case, the BPS wave function is now uniquely specified by the electric and magnetic charges of the black hole, as a coherent eigenstate of the Heisenberg symmetries [17]. It can be pushed down to a wave function on the base space $\mathcal{M}$, annihilated by the quantum BPS constraints, by contour integration along the $\mathbb{CP}^1$ fiber (a quaternionic generalization of the Penrose transform, described in [21, 22].)

While the above statements hold on very general grounds, for practical purposes it is important to have a direct handle on the geometry of the twistor space $\mathcal{Z}$ and the Swann space $\mathcal{S}$. In particular, it is necessary to know the Kähler potential on $\mathcal{Z}$ explicitly, since it controls the inner product on the BPS Hilbert space. To compute the Penrose transform of the BPS wave function on $\mathcal{Z}$, one also needs to express the complex coordinates on $\mathcal{Z}$ and $\mathcal{S}$ in terms of the real coordinates on $\mathcal{M}$ arising from the c-map and the complex coordinates on the fibers. This lays the groundwork for a forthcoming study of the radial quantization of BPS black holes [17], and possibly other physical applications.

1.2 Outline

With this motivation in mind, the goal of the present work is to further elucidate the geometry of the twistor space $\mathcal{Z}$ and the Swann space $\mathcal{S}$, and in particular obtain explicit formulae for their respective complex structures and Kähler potentials.

The outline of this paper is as follows. In section 2, we review the construction of the twistor space $\mathcal{Z}$ and Swann space $\mathcal{S}$ on a general quaternionic-Kähler manifold $\mathcal{M}$; their description in terms of a "generalized prepotential" G (not to be confused with the one controlling higher-derivative corrections on the vector multiplet side) in cases when sufficiently many tri-holomorphic isometries are present; and the relation, recently found in [23, 24], between G and the prepotential F in the case when $\mathcal{M}$ is the c-map of a projective special Kähler manifold $\mathcal{M}_s$.

In section 3, we compute the hyperkähler potential χ on $\mathcal{S}$ and the Kähler potential $K_{\mathcal{Z}}$ on $\mathcal{Z}$, by relaxing the SU(2) gauge choice made in [23]; the latter was sufficient for the

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purpose of computing the metric on the quaternionic-Kähler base but unsuitable for our present purposes. In particular, we uncover a simple relation (3.32) between the Kähler potential $K_{\mathcal{Z}}$ and the Hesse potential Σ on $\mathcal{M}_s$, or equivalently the Bekenstein-Hawking entropy of four-dimensional BPS black holes. We also construct the "covariant c-map" (3.43) and "twistor map" (3.55), which relate the complex coordinates on $\mathcal{S}$ or $\mathcal{Z}$, adapted to the Heisenberg symmetries, to the real coordinates on $\mathcal{M} \times (\mathbb{R}^4/\mathbb{Z}_2)$ or $\mathcal{M} \times \mathbb{CP}^1$, respectively.

In section 4, we apply these techniques to find the general solution for BPS geodesic motion on the c-map manifold $\mathcal{M}$; this is relevant to the problem of constructing spherically symmetric BPS black holes or instantons in $\mathcal{N} = 2$ supergravity. While the physical results obtained are not new, this exercise illustrates the power of the twistor formalism, uncovers the algebraic geometry behind these BPS configurations, and provides a physical explanation for the relation between $K_{\mathcal{Z}}$ and the black hole entropy.

In section 5, we use the twistor map found in section 3 to give a fully explicit integral representation (5.25) of the quaternionic Penrose transform, which relates elements of $H^1(\mathcal{Z}, \mathcal{O}(-k))$ to functions on the quaternionic-Kähler base $\mathcal{M}$, satisfying certain massless field equations. We also find the relevant inner product (5.29), under some assumptions that we spell out. As an example, we compute the Penrose transform (5.35) of an eigenmode (5.30) of the Heisenberg group with vanishing central character on $\mathcal{Z}$. As will be argued in [17], this is the exact radial wave function for a BPS black hole with fixed electric and magnetic charges, in the two-derivative supergravity approximation.

Finally, some additional formulae and derivations used in the main text are given in an appendix at the end of this paper.

Many of the results in this paper were first observed by studying c-maps based on Hermitian symmetric tube domains. As we preview in section 3.3, the corresponding twistor spaces provide a transparent geometric realization of certain group representations constructed in [25], which will be discussed in a separate paper [26].

1.3 Notation

For the reader's convenience, we collect here and in table 1 some notation used (and defined) at various places throughout the paper. Throughout the paper $\mathcal{M}$ is a quaternionic-Kähler manifold of real dimension $4n$. Except in sections 2.1, 2.2 and 5.1, $\mathcal{M}$ is obtained by the c-map from a special Kähler manifold $\mathcal{M}_s$. The table summarizes the various spaces related to $\mathcal{M}$ and the coordinate systems used in the paper. The range of the indices are $a \in \{1, \ldots, n-1\}$, $\Lambda \in \{0\} \cup \{a\} = \{0, \ldots, n-1\}$, $I \in \{\flat\} \cup \{\Lambda\} = \{\flat, 0, \ldots, n-1\}$, $A' \in \{1, 2\}$, $A \in \{1, \ldots, 2n\}$. We use generic coordinates x^μ on $\mathcal{M}$ to lighten the notation in statements not depending on a particular coordinate system; similarly $(u^i, \bar{u}^i)$ and $(z^\aleph, \bar{z}^{\bar{\aleph}})$ denote generic complex coordinates for $\mathcal{Z}$ and $\mathcal{S}$. We sometimes drop indices inside arguments of functions, e.g. we write the Kähler potential on $\mathcal{Z}$ as $K_{\mathcal{Z}}(u, \bar{u})$. We emphasize that $z, z^a, z^\aleph$ are all unrelated, as are x^I, x^μ and ζ, ζ^Λ (ζ is a coordinate on the twistor space of $\mathcal{S}$, introduced in section 2.2).

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Notation	Space	Real dim	Coordinate systems
$\mathcal{M}_s$	special Kähler manifold	$2n-2$	$(z^a, \bar{z}^a)$
$\mathcal{M}$	quaternionic-Kähler manifold (c-map of $\mathcal{M}_s$)	$4n$	$(U, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma, z^a, \bar{z}^a)$ (x^μ)
$\mathcal{Z}$	complex contact manifold (twistor space of $\mathcal{M}$)	$4n+2$	$(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha, \bar{\xi}^\Lambda, \bar{\tilde{\xi}}_\Lambda, \bar{\alpha})$ $(x^\mu, z, \bar{z})$ $(u^i, \bar{u}^i)$
$\mathcal{J}$	3-Sasakian manifold (S^3 bundle over $\mathcal{M}$)	$4n+3$	$(u^i, \bar{u}^i, \phi)$
$\mathcal{S}$	hyperkähler manifold (Swann space of $\mathcal{M}$)	$4n+4$	$(v^I, \bar{v}^I, x^I, \theta_I)$ $(v^I, \bar{v}^I, w_I, \bar{w}_I)$ $(u^i, \lambda, \bar{u}^i, \bar{\lambda})$ $(x^\mu, \pi^{A'}, \bar{\pi}^{A'})$ $(z^\aleph, \bar{z}^{\bar{\aleph}})$

Table 1: Overview of the manifolds discussed in the paper and their coordinate systems.

2. Projective superspace description of the c-map: a review

In this section, we review the projective superfield description of the c-map, first obtained in [23, 24]. In section 2.1, we recall some standard facts about the geometry of quaternionic Kähler manifolds, their Swann space and twistor space. In section 2.2, we review the construction of the metric on $\mathcal{S}$ in the tensor multiplet formalism, in the case where $\mathcal{S}$ admits $n+1$ commuting triholomorphic isometries. Finally, in section 2.3, we specialize to the case where $\mathcal{M}$ is obtained via the c-map from a special Kähler manifold. In this case we review the relation between the "generalized prepotential" G entering the tensor multiplet construction and the prepotential F on the special Kähler base.

2.1 Geometry of quaternionic-Kähler manifolds

In this subsection, we collect some standard results about the geometry of quaternionic-Kähler manifolds. Most of these facts can be found in [18, 19] or inferred from these references.

A quaternionic-Kähler manifold $\mathcal{M}$ is a Riemannian manifold of real dimension $4n$ with holonomy group $USp(2n)USp(2) = (USp(2n) \times USp(2))/\mathbb{Z}_2$. The complexified tangent bundle of such an $\mathcal{M}$ splits locally as

$$T_{\mathbb{C}}\mathcal{M} = E \otimes H\,, \tag{2.1}$$

where E and H are complex vector bundles of respective dimensions $2n$ and 2. This decomposition is preserved by the Levi-Civita connection. Hence after choosing local frames for E and H one can trade the vector index μ in $T_{\mathbb{C}}\mathcal{M}$ for a pair of indices AA', where $A \in \{1, \ldots, 2n\}$ and $A' \in \{1, 2\}$. Concretely this is accomplished by contracting with the "quaternionic vielbein", a covariantly constant matrix of one-forms $V^{AA'} = V^{AA'}_\mu \mathrm{d}x^\mu$.

We will sometimes convert between μ and AA' without writing V explicitly. V satisfies a pseudo-reality condition

$$(V^{AA'})^* = \epsilon_{AB}\epsilon_{A'B'}V^{BB'}\,. \tag{2.2}$$

Here $\epsilon_{AB} = \epsilon_{[AB]}$ and $\epsilon_{A'B'} = \epsilon_{[A'B']}$ are covariantly constant tensors in $\wedge^2(E)$ and $\wedge^2(H)$ respectively, which we use to raise and lower the A, A' indices. We will always choose local frames in H such that $\epsilon_{12} = 1$ and the Hermitean metric is $\eta_{A'\bar{B}'} = \delta_{A'}^{B'}$, and denote the corresponding coordinates in the fiber of H by $\pi^{A'}$. Similarly, our frames in E will always be orthonormal, $\eta_{A\bar{B}} = \delta_A^B$.

The spin connection 1-form on $\mathcal{M}$ splits into

$$\Omega_{AA';BB'} = \epsilon_{AB}\, p_{A'B'} + \epsilon_{A'B'}\, q_{AB}\,, \tag{2.3}$$

where $q_{AB} = q_{(AB)}$ and $p_{A'B'} = p_{(A'B')}$ are connection 1-forms valued respectively in $\mathfrak{usp}(2n) \subset S^2(E)$ and $\mathfrak{usp}(2) \subset S^2(H)$. From the quaternionic vielbein one can construct the metric as well as a triplet $\omega^{A'B'} = \omega^{(A'B')}$ of 2-forms:

$$\mathrm{d}s^2_{\mathcal{M}} = \epsilon_{AB}\,\epsilon_{A'B'}\,V^{AA'}\,V^{BB'}\,, \qquad \omega^{A'B'} = \frac{1}{2}\,\epsilon_{AB}\left(V^{AA'}\wedge V^{BB'} + V^{AB'}\wedge V^{BA'}\right)\,. \tag{2.4}$$

If $\mathcal{M}$ were hyperkähler , the $\omega^{A'B'}$ would be the Kähler forms for the three complex structures, and in particular they would be separately closed. In the quaternionic-Kähler case the triplet is covariantly closed with respect to the $USp(2)$ connection:

$$\mathrm{d}\omega^{A'}_{B'} + p^{A'}_{C'}\wedge\omega^{C'}_{B'} - p^{C'}_{B'}\wedge\omega^{A'}_{C'} = 0\,. \tag{2.5}$$

Moreover, the $USp(2)$ curvature is proportional to $\omega^{A'}_{B'}$:

$$\mathrm{d}p^{A'}_{B'} + p^{A'}_{C'}\wedge p^{C'}_{B'} = \frac{\nu}{2}\omega^{A'}_{B'}\,. \tag{2.6}$$

Quaternionic-Kähler manifolds are always Einstein; the constant ν appearing in (2.6) is related to the scalar curvature by $R = 4n(n+2)\nu$, see e.g. [27]. We shall restrict to the negative curvature case (this is always the case for the manifolds appearing in sigma models coupled to $\mathcal{N} = 2$ supergravity [28]).

By contracting $\omega^{A'B'}$ with the metric one obtains the quaternionic structure operators $J^{A'B'}$. These satisfy the quaternionic algebra and are covariantly constant with respect to the $USp(2)$ connection, but do not have vanishing Nijenhuis tensor (see e.g. appendix B in [27]).

There is a standard way to construct a hyperkähler manifold of dimension $4n+4$, fibered over $\mathcal{M}$: namely, the total space $\mathcal{S}$ of $H^\times/\mathbb{Z}_2$ over $\mathcal{M}$, where $H^\times$ is the $\mathbb{C}^2$ bundle H with the zero section deleted, and $\mathbb{Z}_2$ acts as $\pi^{A'} \to -\pi^{A'}$ on the fiber of H. $\mathcal{S}$ is known as the "Swann space" or "hyperkähler cone" of $\mathcal{M}$ [19, 20]. Its metric is

$$\mathrm{d}s^2_{\mathcal{S}} = |\mathrm{D}\pi^{A'}|^2 + \frac{\nu}{4}\,r^2\mathrm{d}s^2_{\mathcal{M}}\,. \tag{2.7}$$

Here, $r^2 \equiv |\pi^1|^2 + |\pi^2|^2$ is the $USp(2)$ invariant norm in the fiber of H, and

$$\mathrm{D}\pi^{B'} \equiv \mathrm{d}\pi^{B'} + p^{B'}_{C'}\pi^{C'} \tag{2.8}$$

is the covariant differential of $\pi^{A'}$. For $\nu < 0$, the case of interest in this paper, the metric (2.7) has indefinite signature $(4,4n)$. In [19] it is shown that it is hyperkähler , with hyperkähler potential (a simultaneous Kähler potential for all complex structures)[1]

$$\chi(x,\pi^{A'},\bar{\pi}^{A'}) = r^2\,. \tag{2.9}$$

The metric (2.7) admits a SU(2) group of isometries, acting on the $\mathbb{R}^4/\mathbb{Z}_2$ fiber by

$$\delta\pi^{A'} = \mathrm{i}\epsilon_3\pi^{A'} + \epsilon_-\bar{\pi}^{A'}\,, \quad \delta\bar{\pi}^{A'} = -\mathrm{i}\epsilon_3\bar{\pi}^{A'} + \epsilon_+\pi^{A'}\,, \tag{2.10}$$

and a homothetic Killing vector ∂_r, equal to the gradient of the hyperkähler potential $\chi = r^2$. It may also be written as

$$\mathrm{d}s^2_{\mathcal{S}} = |\mathrm{D}\pi^{A'}|^2 + \frac{\nu}{2}\,|\pi_{B'}V^{BB'}|^2 \tag{2.11}$$

reflecting one of the complex structures on $\mathcal{S}$, for which $\mathrm{D}\pi^{A'}$ ($A' \in \{1,2\}$) and $\pi_{B'}V^{BB'}$ ($B \in \{1,\dots,2n\}$) are of type $(1,0)$ [18], and the Kähler form is

$$\omega_{\mathcal{S}} = \mathrm{i}\left(\mathrm{D}\pi^{A'} \wedge \mathrm{D}\bar{\pi}_{A'} + \frac{\nu}{2}\,\pi_{A'}\bar{\pi}_{B'}\omega^{A'B'}\right). \tag{2.12}$$

In this paper we will always use this complex structure on $\mathcal{S}$. The other complex structures are obtained by rotating this one under SU(2); their respective Kähler forms can be obtained by taking the real and imaginary part of the (2,0) form

$$\Omega = \mathrm{D}\pi^{A'} \wedge \mathrm{D}\pi_{A'} + \frac{\nu}{2}\,\pi_{A'}\pi_{B'}\omega^{A'B'}\,. \tag{2.13}$$

Ω is not manifestly holomorphic, but indeed defines a holomorphic symplectic structure on $\mathcal{S}$. There is also a natural holomorphic Liouville form

$$\mathcal{X} = \pi_{A'}\mathrm{D}\pi^{A'}\,, \tag{2.14}$$

which obeys (using (2.6))

$$\mathrm{d}\mathcal{X} = \Omega\,, \qquad \iota_{\mathcal{E}}\Omega = 2\mathcal{X}\,, \tag{2.15}$$

where $\mathcal{E}$ is the "Euler" vector field

$$\mathcal{E} = \pi^{A'}\frac{\partial}{\partial\pi^{A'}}\,. \tag{2.16}$$

The cross-section of $\mathcal{S}$ at a fixed value of the hyperkähler potential defines a 3-Sasakian space $\mathcal{J}$, which is a S^3 fiber bundle over the quaternionic-Kähler space $\mathcal{M}$. It is useful to view S^3 as a Hopf fibration over S^2, and choose coordinates

$$\mathrm{e}^{\mathrm{i}\phi} = \sqrt{\pi^2/\bar{\pi}^2}\,, \quad z = \pi^1/\pi^2\,, \tag{2.17}$$

on the U(1) fiber and S^2 base respectively. The metric (2.7) can then be rewritten as

$$\mathrm{d}s^2_{\mathcal{S}} = \mathrm{d}r^2 + r^2\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}}\right] \tag{2.18}$$

[1]This formula holds for quaternionic-Kähler spaces with positive scalar curvature, but can easily be continued to the case of negative scalar curvature by flipping the sign of r^2 [29].

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where σ_i are a triplet of 1-forms,

$$\sigma_1 + \mathrm{i}\sigma_2 = \frac{\mathrm{d}z + \mathcal{P}}{1 + \bar{z}z}, \quad \sigma_3 = \mathrm{d}\phi - \frac{\mathrm{i}}{2(1+z\bar{z})}\left(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z}\right) - \frac{\mathrm{i}}{r^2}\pi^{A'} p_{A'}^{B'} \bar{\pi}_{B'}, \tag{2.19}$$

and $\mathcal{P}$ is the "projectivized" $USp(2)$ connection,

$$\mathcal{P} = p_2^1 + z(p_1^1 - p_2^2) - z^2 p_1^2. \tag{2.20}$$

In (2.18), the term in brackets is the metric on $\mathscr{J}$.

By dividing out the U(1) action on the fiber of $\mathscr{J}$ one obtains a $(4n+2)$-dimensional space $\mathcal{Z} = \mathscr{J}/\,\mathrm{U}(1)$, the twistor space of $\mathcal{M}$. Since the U(1) action preserves the Kähler form (2.12) on $\mathcal{S}$ and the complex structure on $\mathcal{S}$ relates ∂_ϕ to the homothetic Killing vector ∂_r, $\mathcal{Z}$ can be thought of as a Kähler quotient, $\mathcal{Z} = \mathcal{S}/\!/\,\mathrm{U}(1)$. The Kähler metric is

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|\mathrm{d}z + \mathcal{P}|^2}{(1+z\bar{z})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}}, \tag{2.21}$$

with the Kähler form

$$\omega_{\mathcal{Z}} = \mathrm{i}\left(\frac{(\mathrm{d}z + \mathcal{P}) \wedge (\mathrm{d}\bar{z} + \bar{\mathcal{P}})}{(1+z\bar{z})^2} + \frac{\nu}{2}\frac{\bar{\pi}_{A'}\pi_{B'}}{r^2}\omega^{A'B'}\right). \tag{2.22}$$

A Kähler potential $K_{\mathcal{Z}}$ may be obtained from the hyperkähler potential χ on $\mathcal{S}$ by writing

$$\chi(\lambda, \bar{\lambda}, u, \bar{u}) = |\lambda|^2 \mathrm{e}^{K_{\mathcal{Z}}(u,\bar{u})}, \tag{2.23}$$

where (u^i, λ) are complex coordinates on $\mathcal{S}$, such that u^i are inert under the U(1) action and λ transforms with weight 1.

As is well known, the Kähler quotient $\mathcal{Z} = \mathcal{S}/\!/\,\mathrm{U}(1)$ may also be described as $\mathcal{Z} = \mathcal{S}/\mathbb{C}^\times$; the $\mathbb{C}^\times$ action is the one generated by $\mathcal{E}$, $\pi^A \to \mu\pi^A$. This realizes $\mathcal{Z}$ as a complex manifold equipped with a natural holomorphic line bundle, namely $\mathcal{S}$ itself: we call this line bundle $\mathcal{O}(-2)$, and its k-th power $\mathcal{O}(-2k)$.[2] A (holomorphic) function on $\mathcal{S}$, homogeneous of degree ℓ under $\pi^A \to \mu\pi^A$, is thus a (holomorphic) section of $\mathcal{O}(\ell)$. From this point of view, (2.23) is simply the statement $K_{\mathcal{Z}} = \log\|s\|^2$ where s is a local holomorphic section of $\mathcal{O}(-1)$ and $\|\cdot\|$ is the norm induced from the one in H.

The Liouville form $\mathcal{X}$ on $\mathcal{S}$ descends to a complex contact structure on $\mathcal{Z}$ [18], given by an $\mathcal{O}(2)$-valued holomorphic 1-form X. (Indeed, by definition an $\mathcal{O}(2)$-valued 1-form on $\mathcal{Z}$ is the same as a 1-form on $\mathcal{S}$ which is homogeneous of degree 2 in the $\pi^{A'}$ and has zero inner product with $\mathcal{E}$.) Using (2.15), X may also obtained by contracting Ω with $\frac{1}{2}\mathcal{E}$. To represent it as a 1-form on $\mathcal{Z}$ we must choose some local section of $\mathcal{O}(2)$, thus locally trivializing the line bundle. One natural choice is given by the degree 2 homogeneous (non-holomorphic) function $\pi^1\pi^2$. Then a short computation from (2.14) gives

$$X = \frac{\mathrm{d}z + \mathcal{P}}{z}. \tag{2.24}$$

[2]This notation is justified by the fact that $\mathcal{O}(-2)$ restricts to the usual line bundle $\mathcal{O}(-2)$ over each $\mathbb{CP}^1$ fiber of $\mathcal{Z}$. Locally one can define a bundle $\mathcal{O}(-1)$ as well, namely the total space of $H^\times$ instead of $H^\times/\mathbb{Z}_2$. However, the decomposition $T_{\mathbb{C}}\mathcal{M} = E \otimes H$ need not exist globally over $\mathcal{M}$, so globally $\mathcal{O}(-1)$ need not exist.

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Dually, $\mathcal{Z}$ has a holomorphic $\mathcal{O}(-2)$-valued vector field, the "Reeb vector" Y, determined by the conditions

$$\iota_Y \mathrm{d}X = 0\,, \quad \iota_Y X = 1\,. \tag{2.25}$$

Finally, one may obtain the quaternionic-Kähler manifold $\mathcal{M}$ by projecting the metric (2.21) down to the base, orthogonally to Y and its complex conjugate. The process of going from $\mathcal{S}$ to $\mathcal{M}$ is known in supergravity as the superconformal quotient. An important fact is that isometries of $\mathcal{M}$ compatible with the quaternionic structure lift to holomorphic isometries of $\mathcal{Z}$, and to tri-holomorphic isometries of $\mathcal{S}$. More details on the map between hyperkähler and quaternionic-Kähler spaces can be found in [30, 20, 31].

2.2 Tri-holomorphic isometries and projective superspace

In the last subsection we described how to start from a $4n$-dimensional quaternionic-Kähler space $\mathcal{M}$ and build up its $(4n+4)$-dimensional hyperkähler cone $\mathcal{S}$. Here we consider the special case where $\mathcal{S}$ admits $n+1$ commuting triholomorphic isometries. In this case $\mathcal{S}$ admits a very simple description, which from the physical point of view comes from the duality between hypermultiplets and tensor multiplets in four dimensions, and the existence of the (off-shell) projective superspace formalism for tensor multiplets. We review that description here; in section 3 we will use it to get geometric information about $\mathcal{S}$ and $\mathcal{Z}$.

So suppose $\mathcal{S}$ is a hyperkähler manifold of dimension $4(n+1)$ with $n+1$ commuting triholomorphic isometries. Then the metric is of the "generalized Gibbons-Hawking" form [32, 33],

$$\begin{aligned}\mathrm{d}s^2_{\mathcal{S}} =& \mathcal{L}_{x^I x^J}\left(\frac{1}{4}\mathrm{d}x^I \mathrm{d}x^J + \mathrm{d}v^I \mathrm{d}\bar{v}^J\right)\\ &+\frac{1}{4}\mathcal{L}^{x^I x^J}\left(\mathrm{d}\theta_I + \mathrm{i}\mathcal{L}_{v^K x^I}\mathrm{d}v^K - \mathrm{i}\mathcal{L}_{x^I \bar{v}^K}\mathrm{d}\bar{v}^K\right)\left(\mathrm{d}\theta_J + \mathrm{i}\mathcal{L}_{v^L x^I}\mathrm{d}v^L - \mathrm{i}\mathcal{L}_{x^I \bar{v}^L}\mathrm{d}\bar{v}^L\right)\,.\end{aligned} \tag{2.26}$$

In (2.26) we use coordinates $(v^I, \bar{v}^I, x^I, \theta_I)$ on $\mathcal{S}$: v^I is complex and x^I, θ_I are real. $\mathcal{L}$ is a function of $(v^I, \bar{v}^I, x^I)$, known as the "tensor Lagrangian" because of the way it enters the tensor multiplet formalism. We also denoted $\mathcal{L}_{x^I x^J} \equiv \partial_{x^I}\partial_{x^J}\mathcal{L}$ etc, and use $\mathcal{L}^{x^I x^J}$ for the inverse matrix to $\mathcal{L}_{x^I x^J}$.

The requirement that (2.26) is hyperkähler , and moreover that it has a homothetic Killing vector, leads to constraints on $\mathcal{L}$ [20]: $\mathcal{L}$ must be homogeneous of degree 1 in $(v^I, \bar{v}^I, x^I)$, and invariant under a common phase rotation $v^I \to e^{\mathrm{i}\vartheta}v^I$. Furthermore, $\mathcal{L}$ must satisfy a set of linear partial differential equations, given as eqs. (5.10) in [20]. Any solution of these constraints may be expressed as a contour integral

$$\mathcal{L}(v^I, \bar{v}^I, x^I) = \mathrm{Im}\oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\, G(\eta^I(\zeta))\,, \tag{2.27}$$

where η^I are "real $\mathcal{O}(2)$ projective superfields", written

$$\eta^I = \frac{v^I}{\zeta} + x^I - \bar{v}^I\zeta\,, \tag{2.28}$$

and $G(\eta^I)$ is a holomorphic function, homogeneous of degree 1 in its arguments, which we call the "generalized prepotential". The function G and contour $\mathcal{C}$ completely specify the local hyperkähler geometry of $\mathcal{S}$.

ζ can be thought of as a stereographic coordinate on the $\mathbb{CP}^1$ fiber of the twistor space over $\mathcal{S}$ [33]. As we shall see, at least for the cases we study in the next section, after evaluating the contour integral by residues, ζ becomes identified up to a phase with a natural stereographic coordinate z on the twistor space $\mathcal{Z}$ over $\mathcal{M}$.

To make one of the complex structures of $\mathcal{S}$ explicit, one can trade the real coordinates (x^I, θ_I) for the complex coordinates

$$w_I = \frac{1}{2}\left(\mathcal{L}_{x^I} + \mathrm{i}\theta_I\right) . \tag{2.29}$$

Then the tri-holomorphic isometries $\theta_I \to \theta_I + \epsilon_I$ correspond to imaginary shifts of w_I. The metric (2.26) is hyperkähler , and its hyperkähler potential is the Legendre transform of $\mathcal{L}$ with respect to all x^I, obtained by first computing

$$\chi(v^I, \bar{v}^I, \chi_I) \equiv \mathcal{L}(v, \bar{v}, x) - \chi_I\, x^I , \qquad \chi_I = \frac{\partial \mathcal{L}}{\partial x^I} . \tag{2.30}$$

and then substituting $\chi_I = w_I + \bar{w}_I$ to obtain $\chi(v^I, w_I, \bar{v}^I, \bar{w}_I)$. The hyperkähler cone corresponds to an open domain in the space $\mathbb{R}^{4n+4}$ spanned by the variables v^I, w_I, bounded by the tip of the cone $\chi = 0$.

χ is a function of $(v^I, \bar{v}^I, w_I + \bar{w}_I)$ only, and has scaling weight 2 under the $\mathbb{R}^\times$ action

$$v^I \to \mu^2 v^I , \quad w_I \to w_I . \tag{2.31}$$

Moreover, it is invariant under SU(2) transformations acting on (v^I, w_I) as [20]

$$\delta v^I = \mathrm{i}\epsilon^3 v^I + \epsilon^+ x^I , \quad \delta\bar{v}^I = -\mathrm{i}\epsilon^3 \bar{v}^I + \epsilon^- x^I , \quad \delta w_I = \epsilon^+ \mathcal{L}_{v^I} , \quad \delta\bar{w}_I = \epsilon^- \mathcal{L}_{\bar{v}^I} , \tag{2.32}$$

where x^I is related to $(v^I, w_I, \bar{v}^I, \bar{w}_I)$ by the inverse Legendre transform,

$$x^I = \frac{\partial \chi}{\partial \chi_I} . \tag{2.33}$$

These transformations reflect the fact that $\vec{r}^I = (r^3, r^+, r^-)^I = (x^I, 2v^I, 2\bar{v}^I)$ transforms linearly as a three-vector,

$$\delta x^I = -2(\epsilon^- v^I + \epsilon^+ \bar{v}^I) , \quad \delta v^I = \mathrm{i}\epsilon^3 v^I + \epsilon^+ x^I , \quad \delta\bar{v}^I = -\mathrm{i}\epsilon^3 \bar{v}^I + \epsilon^- x^I . \tag{2.34}$$

The holomorphic symplectic and Liouville forms on $\mathcal{S}$ take the simple form [20]

$$\Omega = \mathrm{d}w_I \wedge \mathrm{d}v^I , \quad \mathcal{X} = v^I\, \mathrm{d}w_I , \tag{2.35}$$

so (v^I, w_I) can be thought of as holomorphic Darboux coordinates for $\mathcal{S}$.

As described in section 2.1, the twistor space $\mathcal{Z}$ can be obtained by a $\mathbb{C}^\times$ quotient of $\mathcal{S}$. In the coordinates (v^I, w_I) the relevant $\mathbb{C}^\times$ action is just complex multiplication on all

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the v^I [20]. So we can single out one coordinate, say $v^\flat$, and define coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ (collectively denoted as u_i in section 2.1) on $\mathcal{Z}$ by

$$v^\Lambda = v^\flat \xi^\Lambda\,, \quad w_\Lambda = \frac{\mathrm{i}}{2}\tilde{\xi}_\Lambda\,, \quad w_\flat = \frac{1}{4\mathrm{i}}\left(\alpha + \xi^\Lambda \tilde{\xi}_\Lambda\right)\,. \tag{2.36}$$

In addition, $\lambda^2 = v^\flat$ defines a local trivialization of $\mathcal{O}(-2)$ over $\mathcal{Z}$. By homogeneity, the hyperkähler potential χ factorizes as in (2.23),

$$\chi = \sqrt{v^\flat \bar{v}^\flat}\, \mathrm{e}^{K_{\mathcal{Z}}(u,\bar{u})}\,. \tag{2.37}$$

We will use this relation in section 3.3 to determine $K_{\mathcal{Z}}$.

Finally, expressing the holomorphic symplectic form Ω in terms of $v^\flat$ and $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$, and contracting with $\partial_{v^\flat}$, gives the holomorphic contact form on $\mathcal{Z}$ and its associated Reeb vector [20],

$$X = 2(\mathrm{d}\alpha + \tilde{\xi}^\Lambda \mathrm{d}\xi_\Lambda - \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda)\,, \quad Y = \frac{1}{2}\frac{\partial}{\partial\alpha}\,. \tag{2.38}$$

These expressions are valid in the holomorphic trivialization $v^\flat = 1$, and motivated the introduction of the coordinates in (2.36).

2.3 c-map spaces and their generalized prepotentials

We now specialize to the case where the quaternionic-Kähler manifold $\mathcal{M}$ arises by applying the c-map to a projective (i.e. non-rigid) special Kähler manifold $\mathcal{M}_s$.

First recall that locally the geometry of $\mathcal{M}_s$ is determined by a single holomorphic function $F(X^\Lambda)$, homogeneous of degree two ($\Lambda \in \{0, \dots, n-1\}$). Namely, choosing a symplectic section $(X^\Lambda(z), F_\Lambda(X(z)) \equiv \partial F/\partial X^\Lambda)$ over $\mathcal{M}_s$, the metric in $\mathcal{M}_s$ is

$$\mathcal{G}_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}(X(z), \bar{X}(\bar{z}))\,, \tag{2.39a}$$

$$\mathcal{K}(X, \bar{X}) \equiv -\log K(X, \bar{X})\,, \tag{2.39b}$$

$$K(X, \bar{X}) \equiv X^\Lambda N_{\Lambda\Sigma} \bar{X}^\Sigma\,, \tag{2.39c}$$

where $N_{\Lambda\Sigma}(X, \bar{X})$ is given by the usual special geometry formula

$$N_{\Lambda\Sigma} \equiv \mathrm{i}\left(F_{\Lambda\Sigma} - \bar{F}_{\Lambda\Sigma}\right)\,, \qquad F_{\Lambda\Sigma} \equiv \partial_{X^\Lambda}\partial_{X^\Sigma} F\,. \tag{2.40}$$

Now we define $\mathcal{M}$ locally as an $\mathbb{R}^{2n+1}$ bundle over $\mathbb{R}^\times \times \mathcal{M}_s$: the fiber is parameterized by $2n+1$ real coordinates $(\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$, the $\mathbb{R}^\times$ factor by a real coordinate e^U. The quaternionic-Kähler metric on $\mathcal{M}$ is then [1, 2]

$$\mathrm{d}s^2_{\mathcal{M}} = 4\,\mathrm{d}U^2 - \mathrm{e}^{-2U} (\mathcal{N} + \bar{\mathcal{N}})_{\Lambda\Sigma} W^\Lambda \bar{W}^\Sigma + \frac{1}{16}\,\mathrm{e}^{-4U}\left(\mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda\right)^2 + 4\,\mathcal{G}_{a\bar{b}}\,\mathrm{d}z^a \mathrm{d}\bar{z}^{\bar{b}} \tag{2.41}$$

where we defined

$$\mathcal{N}_{\Lambda\Sigma} \equiv -\mathrm{i}\bar{F}_{\Lambda\Sigma} - \frac{(NX)_\Lambda (NX)_\Sigma}{(XNX)}\,, \quad W^\Lambda \equiv (\mathcal{N} + \bar{\mathcal{N}})^{-1\,\Lambda\Sigma}\left(\bar{\mathcal{N}}_{\Sigma\Pi} \mathrm{d}\zeta^\Pi + \mathrm{i}\mathrm{d}\tilde{\zeta}_\Sigma\right). \tag{2.42}$$

This space admits isometries acting on the $\mathbb{R}^{2n+1}$ fiber,

$$P^\Lambda = \partial_{\tilde{\zeta}_\Lambda} - \zeta^\Lambda \partial_\sigma\,, \quad Q_\Lambda = -\partial_{\zeta^\Lambda} - \tilde{\zeta}_\Lambda \partial_\sigma\,, \quad K = \partial_\sigma\,, \tag{2.43}$$

which satisfy the Heisenberg algebra

$$[P^\Lambda, Q_\Sigma] = -2\,\delta^\Lambda_\Sigma\, K\,. \tag{2.44}$$

Physically, the metric (2.41) describes the classical moduli space of the $D = 3$ theory obtained by beginning with a $D = 4$, $\mathcal{N} = 2$ supergravity theory coupled to $n-1$ Abelian vector multiplets, then reducing the theory along a spacelike direction. In this context, the scalars z^a are the moduli of the $D = 4$ theory (e.g. Kähler or complex structure moduli for Type IIA or IIB respectively compactified on a Calabi-Yau threefold), e^U is the radius of the fourth direction, ζ^Λ are the fourth component of the gauge fields, $\tilde{\zeta}_\Lambda$ are the Poincaré duals of the $D = 3$ one-forms coming from the reduction of the vector fields in 4 dimensions, and σ is the dual of the Kaluza-Klein connection. Worldsheet instantons in general break the isometries (2.43).

The same type of metric also occurs as the tree-level hypermultiplet moduli space already in 4 dimensions, with a different interpretation of the coordinates: now z^a are complex structure or Kähler moduli respectively in Type IIA or IIB, $\mathrm{e}^{2U} = \mathrm{e}^\phi$ is the four-dimensional dilaton, and $(\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$ are scalars coming from the Ramond-Ramond sector. Space-time instantons are now responsible for the breaking of the isometries (2.43). T-duality along the fourth dimension exchanges these two descriptions of the moduli space.

Finally, the analytic continuation $(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \to \mathrm{i}(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ of the metric (2.41), which arises from the reduction of $D = 4$, $\mathcal{N} = 2$ supergravity coupled to $n-1$ Abelian vector multiplets along a *timelike* direction, is relevant to the study of stationary black hole solutions [10]. This pseudo-Riemannian manifold, dubbed the c^*-map of $\mathcal{M}_s$ in [10], is of a type called "para-quaternionic-Kähler " in the mathematical literature (see e.g. [34] for a recent review, and [11] for an extensive discussion of the rigid c^*-map).

Having defined $\mathcal{M}$ we now turn to its description in the tensor multiplet formalism. In [23, 24] (see also [35, 36]), it was shown that the quaternionic-Kähler metric (2.41) is determined by a generalized prepotential

$$G(\eta^I) = \frac{F(\eta^\Lambda)}{\eta^\flat}\,, \tag{2.45}$$

where $F(\eta^\Lambda)$ is a prepotential for the special Kähler manifold $\mathcal{M}_s$. Here, the indices $\Lambda \in \{0, \ldots, n-1\}$ as usual in special geometry, while the indices I take one extra value, $I \in \{\flat, 0, \ldots, n-1\}$. Physically, the projective superfield $\eta^\flat$ describes the compensating tensor multiplet used in superconformal calculus (see appendix A). Geometrically, it provides the extra quaternionic variable which extends the quaternionic-Kähler $\mathcal{M}$ to its Swann space $\mathcal{S}$. In order to prove (2.45), the authors of [23, 24] evaluated the Legendre transform (2.30) in a particular SU(2) gauge, where

$$v^\flat = 0\,. \tag{2.46}$$

After performing the superconformal quotient, they found agreement with the metric (2.41) upon identifying[3]

$$X^\Lambda = \frac{v^\Lambda}{\sqrt{x^\flat}}\,, \quad e^{2U} = \frac{K(X,\bar{X})}{4x^\flat}\,, \tag{2.47a}$$

$$\zeta^\Lambda = \frac{x^\Lambda}{x^\flat}\,, \qquad \tilde{\zeta}_\Lambda = -\mathrm{i}(w_\Lambda - \bar{w}_\Lambda) - \frac{1}{2}(F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma})\zeta^\Sigma\,, \tag{2.47b}$$

$$\sigma = 2\mathrm{i}(w_\flat - \bar{w}_\flat) + \zeta^\Lambda \tilde{\zeta}_\Lambda + \frac{1}{2}\zeta^\Lambda (F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma})\zeta^\Sigma\,. \tag{2.47c}$$

Moreover, the hyperkähler potential in the limit (2.46) was found to be

$$\chi(v^I, \bar{v}^I, w_I, \bar{w}_I) = \sqrt{2\ K(v^\Lambda, \bar{v}^\Lambda)\ \left[(w+\bar{w})_\Lambda N^{\Lambda\Sigma}(w+\bar{w})_\Sigma - (w+\bar{w})_\flat\right]}\,, \tag{2.48}$$

and could be rewritten in a much simpler way as

$$\chi(v,\bar{v},w,\bar{w}) = K\left[X^\Lambda(v,\bar{v},w,\bar{w}), \bar{X}^\Lambda(v,\bar{v},w,\bar{w})\right]. \tag{2.49}$$

It should be stressed that the gauge-fixing (2.46) is only suitable for the purpose of evaluating the metric on the base: it cannot be used directly to obtain the metric on the hyperkähler cone or on the twistor space. In the next section, we repeat the analysis of [23], without making the gauge choice (2.46).

3. Kähler potentials, covariant c-map and twistor map

In this section, we apply the recipe outlined in section 2.2 to the generalized prepotential (2.45). In sections 3.1, 3.2 and 3.3, we evaluate the contour integral (2.27), take its Legendre transform, and obtain the hyperkähler potential χ on the hyperkähler cone $\mathcal{S}$, as well as the Kähler potential $K_\mathcal{Z}$ on the twistor space $\mathcal{Z}$. In section 3.4, we perform the superconformal quotient from $\mathcal{S}$ to $\mathcal{M}$, and identify the real coordinates on $\mathcal{M}$ as $\mathbb{R}^\times \times \mathrm{SU}(2)$ functions on $\mathcal{S}$; we refer to the relations (3.43) as the "covariant c-map". Some of the technical details of the derivation are presented in appendix A. In section 3.5, we work out the converse, and express the complex coordinates on $\mathcal{S}$ and $\mathcal{Z}$ in terms of the real coordinates on $\mathcal{M} \times \mathbb{R}^4$ and $\mathcal{M} \times S_2$, respectively; we refer to the relations (3.55) as the "twistor map".

3.1 The tensor Lagrangian

We start by evaluating the tensor Lagrangian (2.27) based on the generalized prepotential (2.45), without making any gauge choice. The tensor Lagrangian is the imaginary part of

$$\mathcal{I} = \oint_\mathcal{C} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{F(\eta^\Lambda)}{\eta^\flat} = \oint_\mathcal{C} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta^2}\frac{F(\zeta\eta^\Lambda)}{\zeta\eta^\flat}\,, \tag{3.1}$$

where the contour $\mathcal{C}_+$ is taken to be a small circle around a root ζ_+ of $\zeta\eta^\flat$, given in (2.28) as

$$\zeta\eta^\flat = -\bar{v}^\flat(\zeta-\zeta_+)(\zeta-\zeta_-) \tag{3.2}$$

[3] Compared to [23, 24], we have set $\phi = 2U$, $A^\Lambda = \frac{1}{2}\zeta^\Lambda$, $B_\Lambda = -\tilde{\zeta}_\Lambda$ and multiplied σ by $\frac{1}{4}$.

where

$$\zeta_\pm = \frac{x^\flat \mp r^\flat}{2\bar{v}^\flat}\,, \qquad \zeta_+ - \zeta_- = -\frac{r^\flat}{\bar{v}^\flat}\,, \qquad \zeta_+\zeta_- = -\frac{v^\flat}{\bar{v}^\flat}\,, \tag{3.3}$$

with $r^\flat = \sqrt{(x^\flat)^2 + 4v^\flat\bar{v}^\flat}$. Note in particular that the two roots are antipodes on $\mathbb{CP}^1$:

$$\zeta_+\overline{\zeta_-} = -1\,. \tag{3.4}$$

It will be convenient also to introduce the real quantities

$$C = r^\flat - x^\flat\,, \quad \tilde{C} = r^\flat + x^\flat. \tag{3.5}$$

We can now easily do the contour integral (3.1) and find

$$\mathcal{I} = \frac{F(\eta^\Lambda(\zeta_+))}{r^\flat}\,, \tag{3.6}$$

with

$$\eta^\Lambda(\zeta_\pm) = x^\Lambda - \frac{x^\flat}{2}\Big(\frac{v^\Lambda}{v^\flat} + \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\Big) \pm \frac{r^\flat}{2}\Big(-\frac{v^\Lambda}{v^\flat} + \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\Big)\,, \tag{3.7}$$

which obeys

$$\overline{[\eta^\Lambda(\zeta_+)]} = \eta^\Lambda(\zeta_-)\,. \tag{3.8}$$

Taking the imaginary part of (3.6), we find the tensor Lagrangian

$$\boxed{\mathcal{L}(v,\bar{v},x) = -\tfrac{\mathrm{i}}{2r^\flat}\Big(F(\eta_+^\Lambda) - \bar{F}(\eta_-^\Lambda)\Big)} \tag{3.9}$$

Here and henceforth, we denote $\eta_\pm^\Lambda \equiv \eta^\Lambda(\zeta_\pm)$.

To compute the hyperkähler potential χ we have to take the Legendre transform of (3.9) with respect to x^I. Using the homogeneity of F, which gives

$$2F(\eta_+) = \eta_+^\Lambda F_\Lambda(\eta_+)\,, \qquad F_\Lambda(\eta_+) = \eta_+^\Sigma F_{\Lambda\Sigma}(\eta_+)\,, \tag{3.10}$$

it is an easy exercise to show that (with $K(\eta_+,\eta_-) \equiv \eta_-^\Lambda N_{\Lambda\Sigma}\eta_+^\Sigma$)

$$\begin{aligned}\mathcal{L}(v,\bar{v},x) &= \frac{1}{4r^\flat}K(\eta_+,\eta_-) + \frac{1}{2}(\eta_+^\Lambda + \eta_-^\Lambda)\partial_{x^\Lambda}\mathcal{L} &(3.11)\\ &= x^\Lambda\partial_{x^\Lambda}\mathcal{L} + \frac{1}{4r^\flat}K(\eta_+,\eta_-) - \frac{x^\flat}{2}\left(\frac{v^\Lambda}{v^\flat} + \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\right)\partial_{x^\Lambda}\mathcal{L}\,. &(3.12)\end{aligned}$$

From (3.12) it follows directly that the Legendre transform of $\mathcal{L}$ with respect to x^Λ is

$$\big\langle\mathcal{L} - x^\Lambda\chi_\Lambda\big\rangle_{x^\Lambda} = \frac{1}{4r^\flat}K(\eta_+,\eta_-) - \frac{x^\flat}{2}\left(\frac{v^\Lambda}{v^\flat} + \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\right)\chi_\Lambda\bigg|_{\frac{\partial\mathcal{L}}{\partial x^\Lambda}=\chi_\Lambda}\,, \tag{3.13}$$

where we introduced the "magnetic potential"[4] χ_Λ as the conjugate to x^Λ, and the angle brackets indicate that we evaluate at a critical value of x^Λ.

To finish the Legendre transform computation of the hyperkähler potential χ from $\mathcal{L}$, we would need to transform over the remaining x variable $x^\flat$. In principle one could do this by first expressing the x^Λ as functions of $(v^I,\bar{v}^I,\chi_\Lambda,x^\flat)$, substituting these expressions in $K(\eta_+,\eta_-)$, and then directly computing the transform over $x^\flat$. In the next subsection we will see a more elegant way forward.

[4]This terminology anticipates the relation between χ_Λ and the magnetic charge p^Λ, explained in sections 3.2 and 4.

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3.2 Legendre transform, Hesse potential and black hole entropy

In the last section we introduced the magnetic potentials χ_Λ, determined by extremizing the left side of (3.13) to be

$$\chi_\Lambda = \frac{\partial \mathcal{L}}{\partial x^\Lambda} = -\frac{\mathrm{i}}{2r^\flat}\Big(F_\Lambda(\eta_+) - \bar{F}_\Lambda(\eta_-)\Big). \tag{3.14}$$

It turns out to be convenient (as suggested by symplectic invariance) to introduce as well the "electric potentials"

$$\phi^\Lambda \equiv -\frac{\mathrm{i}}{2r^\flat}\left(\eta_+^\Lambda - \eta_-^\Lambda\right) = \frac{\mathrm{i}}{2}\left(\frac{v^\Lambda}{v^\flat} - \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\right), \tag{3.15}$$

so that $\tilde{\phi}^\Lambda \equiv r^\flat \phi^\Lambda$ and $\tilde{\chi}_\Lambda \equiv r^\flat \chi_\Lambda$ are related to $\eta_\pm^\Lambda$ by

$$\begin{pmatrix} \tilde{\phi}^\Lambda \\ \tilde{\chi}_\Lambda \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \eta_+^\Lambda \\ F_\Lambda(\eta_+) \end{pmatrix}. \tag{3.16}$$

This equation, which determines the complex variable η_+^Λ (at least locally) in terms of the real quantities $(\tilde{\phi}^\Lambda, \tilde{\chi}_\Lambda)$, is familiar from the study of the attractor mechanism in $\mathcal{N}=2$ supergravity [37–39, 8]; namely, in the geometry of a BPS black hole with charges (p^Λ, q_Λ), the properties of the horizon are determined by solving the equations

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X^\Lambda \\ F_\Lambda(X) \end{pmatrix} \tag{3.17}$$

for X^Λ: the moduli at the horizon are given by the ratios of the X^Λ, while the tree-level Bekenstein-Hawking entropy[5] of the black hole is given by

$$K(X, \bar{X}) = 4\,\Sigma(p^\Lambda, q_\Lambda)\,, \tag{3.18}$$

where $\Sigma(p^\Lambda, q_\Lambda)$ is a homogeneous function of degree 2 of the charge vector (p^Λ, q_Λ), invariant under symplectic transformations. The converse of the map (3.17) is then given by

$$X^\Lambda = p^\Lambda + \mathrm{i}\frac{\partial \Sigma(p,q)}{\partial q_\Lambda}\,, \quad F_\Lambda(X) = q_\Lambda - \mathrm{i}\frac{\partial \Sigma(p,q)}{\partial p^\Lambda}\,. \tag{3.19}$$

The function $\Sigma(p^\Lambda, q_\Lambda)$ is also familiar as the Hesse potential of rigid special Kähler geometry, where it determines the metric in real coordinates [40–42].

Applying the above to (3.16), and making use of the homogeneity of F, we find

$$K(\eta_+, \eta_-) = 4(r^\flat)^2\,\Sigma(\phi^\Lambda, \chi_\Lambda) \tag{3.20}$$

and

$$\eta_+^\Lambda = r^\flat\left(\mathrm{i}\phi^\Lambda - \frac{\partial \Sigma(\phi,\chi)}{\partial \chi_\Lambda}\right), \quad F_\Lambda(\eta_+) = r^\flat\left(\mathrm{i}\chi_\Lambda + \frac{\partial \Sigma(\phi,\chi)}{\partial \phi^\Lambda}\right). \tag{3.21}$$

In our computation of the Legendre transform below, the form (3.20) of K will be useful, because all the $x^\flat$ dependence has been isolated in the prefactor.

[5] We have chosen a convenient normalization for Newton's constant, $G_N = \pi$.

3.3 Potentials on hyperkähler cone and twistor space

Substituting the result (3.20) in (3.13), we now consider the remaining Legendre transform with respect to $x^\flat$,

$$\Big\langle \mathcal{L} - x^\Lambda \chi_\Lambda - x^\flat \chi_\flat \Big\rangle_{x^\Lambda, x^\flat} = \Big\langle r^\flat \Sigma(\phi^\Lambda, \chi_\Lambda) - x^\flat \tilde{\chi}_\flat \Big\rangle_{x^\flat} , \tag{3.22}$$

where we defined

$$\tilde{\chi}_\flat \equiv \chi_\flat + \frac{1}{2}\left(\frac{v^\Lambda}{v^\flat} + \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\right)\chi_\Lambda . \tag{3.23}$$

We want to determine the value of $x^\flat$ extremizing the right side. Noting that ϕ^Λ and χ_Λ are independent of $x^\flat$ (the latter by definition of the Legendre transform), we find

$$x^\flat = \pm \frac{2\sqrt{v^\flat \bar{v}^\flat}\,\tilde{\chi}_\flat}{\sqrt{\Sigma^2(\phi,\chi) - \tilde{\chi}_\flat^2}} , \tag{3.24}$$

We assume that the term under the square root is positive definite in the region of interest, and choose the upper sign below; while a solution with the opposite sign does in principle exist, it leads to a much more complicated form of the hyperkähler potential. From (3.24) it easily follows that

$$r^\flat = \frac{2\sqrt{v^\flat \bar{v}^\flat}\,\Sigma(\phi,\chi)}{\sqrt{\Sigma^2(\phi,\chi) - \tilde{\chi}_\flat^2}} . \tag{3.25}$$

We then find that (3.22) simplifies to

$$\Big\langle \mathcal{L} - x^\Lambda \chi_\Lambda - x^\flat \chi_\flat \Big\rangle_{x^\Lambda, x^\flat} = 2\sqrt{v^\flat \bar{v}^\flat \left(\Sigma^2(\phi,\chi) - \tilde{\chi}_\flat^2\right)} . \tag{3.26}$$

As described below (2.30), the hyperkähler potential χ is obtained from this Legendre transform upon replacing χ_I by $w_I + \bar{w}_I$. Inserting as well the definitions (3.15), (3.23) of ϕ^Λ and $\tilde{\chi}_\flat$, we conclude that χ is given in terms of the complex variables (v^I, w_I) by

$$\boxed{\begin{aligned} \chi(v,\bar{v},w,\bar{w}) = 2\sqrt{v^\flat \bar{v}^\flat}\, \Big\{ &\Sigma^2\left[\tfrac{i}{2}\left(\tfrac{v^\Lambda}{v^\flat} - \tfrac{\bar{v}^\Lambda}{\bar{v}^\flat}\right), w_\Lambda + \bar{w}_\Lambda\right] \\ &- \left[w_\flat + \bar{w}_\flat + \tfrac{1}{2}\left(\tfrac{v^\Lambda}{v^\flat} + \tfrac{\bar{v}^\Lambda}{\bar{v}^\flat}\right)(w_\Lambda + \bar{w}_\Lambda)\right]^2 \Big\}^{\frac{1}{2}} \end{aligned}} \tag{3.27}$$

The condition that the term in bracket be strictly positive defines an open set in $\mathbb{R}^{4n+4}$, which we identify as the Swann space of $\mathcal{M}$.

Thus, the hyperkähler potential of the Swann space $\mathcal{S}$ is simply expressed in terms of the Hesse potential Σ, or equivalently the Bekenstein-Hawking entropy functional.[6] This parallels the rigid case, where the Hesse potential Σ is known to provide a Kähler potential for the rigid c-map in complex Darboux coordinates [1, 40, 42].

[6]The relation of χ to the black hole entropy will find a natural physical explanation in section 4, Equation (4.26).

We note that the hyperkähler potential (3.27), and therefore the metric on $\mathcal{S}$, are invariant under the Killing vector fields

$$P^\Lambda = \frac{\mathrm{i}}{2}\,\partial_{w_\Lambda} + \text{c.c.}\,, \quad Q_\Lambda = -v^\flat \partial_{v^\Lambda} + w_\Lambda \partial_{w_\flat} + \text{c.c.}\,, \quad K = -\frac{\mathrm{i}}{4}\,\partial_{w_\flat} + \text{c.c.} \tag{3.28}$$

satisfying the Heisenberg algebra (2.44). Here, P^Λ and K are the $n+1$ triholomorphic isometries afforded by the tensor multiplet description, while Q_Λ are additional triholomorphic isometries following from the invariance of the generalized prepotential (2.45) under transformations [43, 35]

$$\eta^\Lambda \to \eta^\Lambda + \epsilon^\Lambda \eta^\flat\,. \tag{3.29}$$

In section 3 we show that (3.28) descend to the isometries (2.43) of $\mathcal{M}$ under the superconformal quotient.

We may obtain another useful expression for χ by switching back from $w^\flat$ to the tensor multiplet variable $x^\flat$. Namely, using (3.24) and (3.26) gives directly

$$\chi = 4\frac{v^\flat \bar{v}^\flat}{r^\flat}\Sigma(\phi,\chi)\,. \tag{3.30}$$

Using the relation (3.20) between Σ and the Kähler potential on $\mathcal{M}_s$, this can also be written

$$\chi = \frac{v^\flat \bar{v}^\flat}{(r^\flat)^3} K\Big(\eta_+^\Lambda, \eta_-^\Lambda\Big). \tag{3.31}$$

This relation between the geometry of $\mathcal{S}$ and that of the special Kähler manifold $\mathcal{M}_s$ will become useful in section 3.5.

To discuss the twistor space $\mathcal{Z}$, we use the coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ introduced in (2.36). Plugging them into (3.27), one finds the form of (2.37), with

$$\boxed{K_{\mathcal{Z}} = \tfrac{1}{2}\log\left\{\Sigma^2\left[\tfrac{\mathrm{i}}{2}(\xi^\Lambda - \bar{\xi}^\Lambda), \tfrac{\mathrm{i}}{2}(\tilde{\xi}_\Lambda - \bar{\tilde{\xi}}_\Lambda)\right] + \tfrac{1}{16}\left[\alpha - \bar{\alpha} + \xi^\Lambda \bar{\tilde{\xi}}_\Lambda - \bar{\xi}^\Lambda \tilde{\xi}_\Lambda\right]^2\right\} + \log 2} \tag{3.32}$$

So, as for χ, the Kähler potential on the twistor space $\mathcal{Z}$ is simply expressed in terms of the Hesse potential Σ on the special Kähler manifold $\mathcal{M}_s$ (or, rather, its rigidification $\mathcal{M}'_s$). The range of $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ is restricted to the domain where the sign of the bracket in (3.32) is positive: other values do not correspond to points of $\mathcal{Z}$. The triholomorphic isometries (3.28) of $\mathcal{S}$ descend to holomorphic isometries of $\mathcal{Z}$, generated by the vector fields

$$P^\Lambda = \partial_{\tilde{\xi}_\Lambda} - \xi^\Lambda \partial_\alpha + \text{c.c.}\,, \quad Q_\Lambda = -\partial_{\xi^\Lambda} - \tilde{\xi}_\Lambda \partial_\alpha + \text{c.c.}\,, \quad K = \partial_\alpha + \text{c.c.}\,. \tag{3.33}$$

This standard form of the Heisenberg action was an additional motivation for the change of variable (2.36).

At this stage, we note that when the special Kähler space $\mathcal{M}_s$ is a Hermitian symmetric space $G/K = \mathrm{Conf}(J)/\mathrm{Struc}(J) \times \mathrm{U}(1)$, corresponding to the case where the prepotential F is the cubic norm of a Jordan algebra J, the Hesse potential Σ becomes equal to the square root of the quartic invariant of the conformal group $\mathrm{Conf}(J)$. The term in bracket is then recognized as the "quartic light-cone"[7] $\mathcal{N}_4(\xi,\tilde{\xi},\alpha;\bar{\xi},\bar{\tilde{\xi}},\bar{\alpha})$ introduced in [25]; in that work, it was shown that the locus $\mathcal{N}_4=0$ is invariant under the action of a group QConf(J) containing Conf$(J)\times$SU(2) as a subgroup; in fact, $\log\mathcal{N}_4$ changes by Kähler transformations under this action. This implies that the twistor space $\mathcal{Z}$ carries a holomorphic, isometric action of QConf(J), and that the quaternionic-Kähler base is itself a symmetric space QConf(J)/Conf$(J)\times$SU(2). This fact is at the root of the construction of the quaternionic discrete series representations of QConf(J) [44]. In a separate paper [26], the constructions of [25] will be revisited in light of this observation.

3.4 The covariant c-map

Our next task is to construct the projection from the hyperkähler cone $\mathcal{S}$ to the quaternionic-Kähler base $\mathcal{M}$: we will express the coordinates $(U,z^a,\bar{z}^{\bar{a}},\zeta^\Lambda,\tilde{\zeta}_\Lambda,\sigma)$ on $\mathcal{M}$ as $(\mathbb{R}^\times\times\mathrm{SU}(2))$-invariant functions on $\mathcal{S}$. We first list some possible candidates, and then argue that they are indeed equal to the coordinates on $\mathcal{M}$, as determined by the c-map metric (2.41). Further details of the derivation are given in the appendix.

First we construct a candidate for U. The hyperkähler potential χ itself is SU(2) invariant, but has weight 2 under $\mathbb{R}^\times$; it can be made invariant by dividing out by $r^\flat$,

$$e^{2U}\equiv\frac{\chi}{4r^\flat}=\frac{\Sigma^2(\phi,\chi)-\tilde{\chi}_\flat^2}{4\Sigma(\phi,\chi)} \tag{3.34}$$

Next, recall from (2.34) that $\vec{r}^I=(x^I,2v^I,2\bar{v}^I)$ transforms as an SU(2) vector, and has weight 2 under $\mathbb{R}^\times$. Hence we can construct candidates for ζ^Λ by taking SU(2) invariant dot products,

$$\zeta^\Lambda\equiv\frac{1}{(r^\flat)^2}(\vec{r}^\flat\cdot\vec{r}^\Lambda)=\frac{1}{(r^\flat)^2}\left(x^\flat x^\Lambda+2\bar{v}^\flat v^\Lambda+2v^\flat\bar{v}^\Lambda\right). \tag{3.35}$$

Using (3.7), this may also be written as

$$\zeta^\Lambda=\frac{1}{2}\left(\frac{v^\Lambda}{v^\flat}+\frac{\bar{v}^\Lambda}{\bar{v}^\flat}\right)+\frac{x^\flat}{(r^\flat)^2}\mathrm{Re}[\eta_+^\Lambda]=\frac{1}{2}(\xi^\Lambda+\bar{\xi}^\Lambda)+\frac{x^\flat}{(r^\flat)^2}\mathrm{Re}[\eta_+^\Lambda]. \tag{3.36}$$

Next we construct the coordinates z^Λ on $\mathcal{M}_s$. Using (2.34), one may check that under SU(2) transformations one has

$$\delta[\zeta_+\eta_+^\Lambda]=\left(\mathrm{i}\epsilon^3-2\epsilon^-\zeta_+\right)[\zeta_+\eta_+^\Lambda]. \tag{3.37}$$

Since this is just an overall Λ-independent rescaling, we can construct ratios which are SU(2) and scale invariant:

$$z^a\equiv\frac{\eta_+^a}{\eta_+^0},\qquad a=1,\dots,n-1. \tag{3.38}$$

The remaining coordinates $\tilde{\zeta}_\Lambda$ and σ are trickier to obtain. Symplectic covariance suggests considering the "electric" counterpart of (3.36),

$$\tilde{\zeta}_\Lambda\equiv-\mathrm{i}(w_\Lambda-\bar{w}_\Lambda)+\frac{x^\flat}{(r^\flat)^2}\mathrm{Re}[F_\Lambda(\eta_+)]=\frac{1}{2}(\tilde{\xi}_\Lambda-\bar{\tilde{\xi}}_\Lambda)+\frac{x^\flat}{(r^\flat)^2}\mathrm{Re}[F_\Lambda(\eta_+)], \tag{3.39}$$

[7]We are grateful to M. Günaydin for extensive discussions on these constructions.

whose SU(2) invariance can indeed be checked by a somewhat tedious computation. Finally, an even more tedious computation shows that

$$\sigma \equiv 2\mathrm{i}(w_\flat - \bar{w}_\flat) + \mathrm{i}\Big(\frac{v^\Lambda}{v^\flat} w_\Lambda - \frac{\bar{v}^\Lambda}{\bar{v}^\flat}\bar{w}_\Lambda\Big) - \frac{x^\flat}{(r^\flat)^2}\mathrm{Re}\Big(\eta_+^\Lambda\,\tilde{\zeta}_\Lambda - F_\Lambda(\eta_+)\,\zeta^\Lambda\Big) \tag{3.40}$$

$$= \frac{1}{2}(\alpha+\bar{\alpha}) - \frac{x^\flat}{(r^\flat)^2}\mathrm{Re}\Big(\eta_+^\Lambda\,\tilde{\zeta}_\Lambda - F_\Lambda(\eta_+)\,\zeta^\Lambda\Big) \tag{3.41}$$

is also invariant under SU(2).

We claim that the functions $(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$ on $\mathcal{S}$ which we have just constructed give the projection of $\mathcal{S}$ down to $\mathcal{M}$. As a consistency check, it is straightforward to check that with these identifications, the triholomorphic isometries (3.28) on $\mathcal{S}$ descend to the isometries (2.43) on $\mathcal{M}$. A direct proof would involve performing the superconformal quotient construction in our coordinates and checking that the resulting metric matches (2.41). In appendix A we check this explicitly for the components of the metric along $\mathrm{d}\tilde{\zeta}_\Lambda$ and $\mathrm{d}\sigma$; other components are fixed by supersymmetry.

Moreover, given that it was already verified in [23] that the superconformal quotient of $\mathcal{S}$ yields (2.41), and the $v^\flat \to 0$ limit of the coordinate functions were determined there as (2.47), we only need to check that the $v^\flat \to 0$ limit of our coordinates agrees with (2.47). In this limit, one has to leading order in the $v^\flat$-expansion

$$\eta_+^\Lambda = -\frac{v^\Lambda}{v^\flat}x^\flat + x^\Lambda + O(v^\flat)\,, \tag{3.42}$$

while the poles ζ_+ and ζ_- approach 0 and ∞ respectively. It is straightforward to check that the coordinates $(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda)$ defined in this section indeed agree with (2.47), whereas a similar check for $(\tilde{\zeta}_\Lambda, \sigma)$ necessitates taking into account the next-to-subleading term in (3.42), due to the appearance of certain singular terms in the $v^\flat \to 0$ limit. Moreover, (3.31) reduces to (2.49) in this limit.

We conclude that the coordinates $(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$ on the base $\mathcal{M}$ are given in terms of the complex coordinates $(v^I, w_I, \bar{v}^I, \bar{w}_I)$ on the hyperkähler cone $\mathcal{S}$ (or equivalently, the complex variables $\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha, \bar{\xi}^\Lambda, \bar{\tilde{\xi}}_\Lambda, \bar{\alpha}$ on the twistor space $\mathcal{Z}$):

$$\boxed{\begin{array}{c} \mathrm{e}^{2U} = \chi/(4r^\flat) \quad , \qquad z^a = \eta_+^a/\eta_+^0 \\ \zeta^\Lambda = \frac{1}{2}\left(\xi^\Lambda + \bar{\xi}^\Lambda\right) + \frac{x^\flat}{(r^\flat)^2}\mathrm{Re}[\eta_+^\Lambda] \\ \tilde{\zeta}_\Lambda = \frac{1}{2}\left(\tilde{\xi}_\Lambda + \bar{\tilde{\xi}}_\Lambda\right) + \frac{x^\flat}{(r^\flat)^2}\mathrm{Re}(F_\Lambda(\eta_+)) \\ \sigma = \frac{1}{2}\left(\alpha+\bar{\alpha}\right) - \frac{x^\flat}{(r^\flat)^2}\Big(\mathrm{Re}[\eta_+^\Lambda]\tilde{\zeta}_\Lambda - \mathrm{Re}[F_\Lambda(\eta_+)]\zeta^\Lambda\Big) \end{array}} \tag{3.43}$$

We call these relations the covariant c-map formulae.

3.5 The twistor map

So far we have seen that $\mathcal{S}$ defined by (2.45) is fibered over $\mathcal{M}$, and constructed the projection map explicitly, but we have not given any information about the coordinates in the $\mathbb{R}^4/\mathbb{Z}_2$ fiber. In this section we will show that, given a choice of symplectic section

$(X^\Lambda, F_\Lambda(X))$ over the special Kähler space $\mathcal{M}_s$, there is a canonically defined coordinate z in $\mathcal{Z}$, holomorphic on each twistor fiber. Moreover we give formulas relating z and the coordinates in $\mathcal{M}$ to the complex coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ in $\mathcal{Z}$. We refer to this transformation as the "twistor map". We then construct a corresponding coordinate system in $\mathcal{S}$, with two complex fiber coordinates (π^1, π^2), such that $\pi^1/\pi^2 = z$ and $|\pi^1|^2 + |\pi^2|^2 = \chi$. For applications such as the Penrose transform these coordinates are very convenient, as we will see in section 5.

We start by expressing $\eta_\pm^\Lambda$ in terms of ζ^Λ: a straightforward computation using (3.7) and (3.36) gives

$$\eta_\pm^\Lambda = \frac{v^\Lambda}{\zeta_\pm} + x^\Lambda - \bar{v}^\Lambda \zeta_\pm = \frac{v^\Lambda}{\zeta_\pm} + \frac{(r^\flat)^2 \zeta^\Lambda - 2\bar{v}^\flat v^\Lambda - 2v^\flat \bar{v}^\Lambda}{x^\flat} - \bar{v}^\Lambda \zeta_\pm \,. \tag{3.44}$$

This equation can be used to express ξ^Λ in terms of $\eta_\pm^\Lambda$:

$$\xi^\Lambda = \zeta^\Lambda + \frac{1}{(r^\flat)^2} \left(\frac{v^\flat}{\zeta_+} \eta_+^\Lambda - \bar{v}^\flat \zeta_+ \eta_-^\Lambda \right) . \tag{3.45}$$

Now, suppose we choose a symplectic section $(X^\Lambda(z^a), F_\Lambda(z^a))$ over the special Kähler manifold $\mathcal{M}_s$. From (3.38) we know this section is proportional to $(\eta_+^\Lambda, F_\Lambda(\eta_+))$, so there exists $z \in \mathbb{C}^\times$ with

$$\frac{v^\flat}{(r^\flat)^2 \zeta_+} \begin{pmatrix} \eta_+^\Lambda \\ F_\Lambda(\eta_+) \end{pmatrix} = 2\mathrm{i}\, \mathrm{e}^{U + \frac{1}{2}\mathcal{K}(X,\bar{X})} z^{-1} \begin{pmatrix} X^\Lambda \\ F_\Lambda(X) \end{pmatrix} . \tag{3.46}$$

The reason for choosing the complicated prefactors in (3.46) will become clear below. Using the definition (3.34) of U and the relation (3.31), we can rewrite (3.46) as

$$e^{\frac{1}{2}\mathcal{K}(\eta_+, \bar{\eta}_+)} \begin{pmatrix} \eta_+^\Lambda \\ F_\Lambda(\eta_+^\Lambda) \end{pmatrix} = \mathrm{i}\, \frac{\zeta_+}{z} \sqrt{\frac{\bar{v}^\flat}{v^\flat}}\, e^{\frac{1}{2}\mathcal{K}(X,\bar{X})} \begin{pmatrix} X^\Lambda \\ F_\Lambda(X) \end{pmatrix} . \tag{3.47}$$

Then applying $K(\cdot, \bar{\cdot})$ to both sides shows that the modulus of z is equal to that of ζ_+,

$$z\bar{z} = \zeta_+ \bar{\zeta}_+ = \frac{C}{\tilde{C}} \,, \tag{3.48}$$

where C and $\tilde{C}$ were defined in (3.5). Substituting (3.46) and its conjugate in (3.45), and using (3.48), now establishes our first "twistor map" relation:

$$\xi^\Lambda = \zeta^\Lambda + 2\mathrm{i}\, \mathrm{e}^{U + \frac{1}{2}\mathcal{K}(X,\bar{X})} \left(z \bar{X}^\Lambda + z^{-1} X^\Lambda \right) . \tag{3.49}$$

This relation expresses the complex coordinates ξ^Λ on $\mathcal{Z}$ in terms of the coordinates $(X^\Lambda, \bar{X}^\Lambda, U, \zeta^\Lambda)$ on the base $\mathcal{M}$ and a coordinate z in the twistor fiber. The rationale for the choice of prefactors in (3.46) is now clear: the modulus was chosen such that the ratio between the last two terms in (3.45) has modulus $|z|^2$, while the choice of phase ensures that ξ^Λ depends holomorphically on z when the base coordinates are fixed. In other words, the fiber over every point on the base is rationally embedded in $\mathcal{Z}$, a key property of any twistor construction. Changing the symplectic section on $\mathcal{M}$ by $X \to \mathrm{e}^f X$ transforms z by the phase $\mathrm{e}^{\frac{1}{2}(f-\bar{f})}$.

To compute $\tilde{\xi}_\Lambda$, defined in (2.36), we first write $\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda - (2\mathrm{i}w_\Lambda + \tilde{\zeta}_\Lambda)$ and use the relation (3.39) in the last term. Using $w_\Lambda + \bar{w}_\Lambda = \chi_\Lambda$ in (3.14), we then obtain

$$\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + \frac{1}{(r^\flat)^2}\left(\frac{v^\flat}{\zeta_+} F_\Lambda(\eta_+) - \bar{v}^\flat \zeta_+ \bar{F}_\Lambda(\eta_-)\right). \tag{3.50}$$

Eq. (3.47) then enables us to rewrite (3.50) in parallel to (3.49),

$$\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + 2\mathrm{i}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\left(z\bar{F}_\Lambda + z^{-1}F_\Lambda\right). \tag{3.51}$$

Finally, using (3.35), (3.39) and (3.40), one may show that

$$\alpha = \sigma + \zeta^\Lambda \tilde{\xi}_\Lambda - \tilde{\zeta}_\Lambda \xi^\Lambda . \tag{3.52}$$

Together with (3.49) and (3.51), this implies

$$\alpha = \sigma + 2\mathrm{i}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\left(\bar{W}z + Wz^{-1}\right), \tag{3.53}$$

where W is the symplectic invariant combination (or "superpotential")

$$W = F_\Lambda(X)\,\zeta^\Lambda - X^\Lambda \tilde{\zeta}_\Lambda . \tag{3.54}$$

Altogether, (3.49), (3.51), (3.53) provide the general relation between the complex coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ on $\mathcal{Z}$ and the real coordinates on the base $\mathcal{M}$, together with the fiber coordinate z. Since it is one of the main results of this section, we rewrite the twistor map below:

$$\boxed{\begin{array}{rcl} \xi^\Lambda &=& \zeta^\Lambda + 2\mathrm{i}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\left(z\bar{X}^\Lambda + z^{-1}X^\Lambda\right) \\ \tilde{\xi}_\Lambda &=& \tilde{\zeta}_\Lambda + 2\mathrm{i}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\left(z\bar{F}_\Lambda + z^{-1}F_\Lambda\right) \\ \alpha &=& \sigma + 2\mathrm{i}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\left(z\bar{W} + z^{-1}W\right) \end{array}} \tag{3.55}$$

Finally we give a similar coordinate system in the hyperkähler cone $\mathcal{S}$: as discussed in section 2, we want two complex functions π^1, π^2 on $\mathcal{S}$, holomorphic in each fiber of $\mathcal{S}$ over $\mathcal{M}$, defined up to the $\mathbb{Z}_2$ action $(\pi^1, \pi^2) \to (-\pi^1, -\pi^2)$, and obeying $\pi^1/\pi^2 = z$, $\chi = |\pi^1|^2 + |\pi^2|^2$. A pair of coordinates satisfying our requirements is

$$\boxed{\begin{pmatrix} \pi^1 \\ \pi^2 \end{pmatrix} = 2\,\mathrm{e}^U \sqrt{v^\flat}\begin{pmatrix} z^{\frac{1}{2}} \\ z^{-\frac{1}{2}} \end{pmatrix}} \tag{3.56}$$

Indeed, we compute

$$|\pi^1|^2 + |\pi^2|^2 = 4\,\mathrm{e}^{2U}\,|v^\flat|\,\left(|z| + |z|^{-1}\right). \tag{3.57}$$

Using $|z| = |\zeta_+|$ we see that this is $4r^\flat \mathrm{e}^{2U}$, which by (3.34) is equal to χ as desired. With the knowledge of (3.56), we may then translate the holomorphic contact form (2.24) into the holomorphic trivialization $v^\flat = 1$ appropriate for comparison with (2.38),

$$X = 4\,e^{2U}\,\frac{\mathrm{d}z + \mathcal{P}}{z} = \frac{\mathrm{e}^{K_{\mathcal{Z}}}}{1 + z\bar{z}}\sqrt{\frac{\bar{z}}{z}}\,(\mathrm{d}z + \mathcal{P}). \tag{3.58}$$

4. Integrability of the BPS geodesic flow

In this section, we apply twistorial methods to find the general solution for supersymmetric geodesic motion on the quaternionic-Kähler metric (2.41). After suitable analytic continuation, this problem is equivalent to the construction of stationary, spherically symmetric black hole solutions in $\mathcal{N}=2$ supergravity coupled to vector multiplets [10], or spherically symmetric instantons in $\mathcal{N}=2$ supergravity coupled to hypermultiplets [14]. The corresponding solutions (as well as their multi-centered generalizations) have been known explicitly for some time [45 − 48, 15]. We rederive them here to illustrate the power of the twistor formalism, and illuminate the geometric structure behind these supergravity solutions. We expect that similar arguments can be used to generate new solutions in a variety of other contexts where supersymmetry can be reduced to holomorphy.

4.1 Strategy

As will be shown in [17], there is a correspondence between geodesics on a quaternionic-Kähler manifold $\mathcal{M}$ and geodesics on its hyperkähler cone $\mathcal{S}$ with zero angular momentum under the global SU(2). Moreover, BPS geodesic motion on $\mathcal{M}$, characterized by the condition that the quaternionic vielbein $V^{AA'}$ pulled back to the geodesic has a right-eigenvector with eigenvalue zero, is equivalent to holomorphic geodesic motion on $\mathcal{S}$:

$$p_{\aleph} = 0\,, \tag{4.1}$$

where $p_{\aleph}$ denotes the canonical momenta conjugate to the holomorphic coordinates $z^{\aleph}$ on $\mathcal{S}$; it will be convenient to choose the holomorphic coordinates $(z^{\aleph}) = (\xi^{\Lambda}, \tilde{\xi}_{\Lambda}, w_{\flat}, v^{\flat})$ on the hyperkähler cone. In particular, (4.1) implies that the geodesic is null,

$$p_{\aleph} g^{\aleph\bar{\aleph}} p_{\bar{\aleph}} = 0\,. \tag{4.2}$$

It is impossible for real non-constant geodesics on $\mathcal{S}$ to satisfy (4.1), since $p_{\bar{\aleph}} = p^*_{\aleph}$. For the analytic continuations of $\mathcal{S}$ relevant to the black hole or instanton problems, however, the BPS conditions can be satisfied. In this section, we take the metric on $\mathcal{S}$ to be the standard metric on the Swann space of the quaternionic-Kähler c-map metric (2.41), but treat the holomorphic and anti-holomorphic coordinates $z^{\aleph}$ and $\bar{z}^{\bar{\aleph}}$ independently, i.e. we work with the complexification of $\mathcal{S}$. We return to the issue of reality conditions at the end of section 4.2.

The BPS condition (4.1) implies that the anti-holomorphic coordinates $(z^{\bar{\aleph}}) = (\bar{\xi}^{\Lambda}, \bar{\tilde{\xi}}_{\Lambda}, \bar{w}_{\flat}, \bar{v}^{\flat})$ are constants of motion. Moreover, the conservation of the Noether charges[8] associated with the Heisenberg and $\mathrm{U}(1) \subset \mathrm{SU}(2)$ symmetries (the latter vanishing for

[8] $(P^{\Lambda}, Q_{\Lambda}, K)$ will in general differ from the charges $(p^{\Lambda}, q_{\Lambda}, k)$ on the base, e.g. due to the rescaling of the metric on $\mathcal{M}$ by $r^2 = \chi$.

geodesics with zero SU(2) momentum),

$$P^\Lambda = p_{\tilde\xi_\Lambda} + p_{\bar{\tilde\xi}_\Lambda}\,, \tag{4.3a}$$

$$Q_\Lambda = -p_{\xi^\Lambda} - p_{\bar\xi^\Lambda} + \frac{\mathrm{i}}{2}(\tilde\xi_\Lambda p_{w_\flat} - \bar{\tilde\xi}_\Lambda p_{\bar w_\flat})\,, \tag{4.3b}$$

$$K = \frac{\mathrm{i}}{4}(p_{w_\flat} - p_{\bar w_\flat})\,, \tag{4.3c}$$

$$0 = \frac{1}{2\mathrm{i}}(v^\flat p_{v^\flat} - \bar v^\flat p_{\bar v^\flat})\,, \tag{4.3d}$$

implies that the anti-holomorphic momenta $p_{\bar\aleph} = (p_{\bar\xi^\Lambda}, p_{\bar{\tilde\xi}_\Lambda}, p_{\bar w_\flat}, p_{\bar v^\flat})$ are also constants of motion (and, moreover, that $p_{\bar v^\flat} = 0$). It turns out that these conserved quantities are sufficient to integrate the motion completely.

Indeed, $p_{\bar\aleph}$ being constant, the first order equation

$$g_{\bar\aleph\aleph}\frac{\mathrm{d}z^\aleph}{\mathrm{d}t} = p_{\bar\aleph} \tag{4.4}$$

can be integrated using the Kähler property $g_{\bar\aleph\aleph} = \partial_{z^{\bar\aleph}}\partial_{z^\aleph}\chi$ of the metric, to give

$$\boxed{\partial_{\bar\aleph}\chi = p_{\bar\aleph}t + c_{\bar\aleph}} \tag{4.5}$$

where $c_{\bar\aleph}$ are constants of integration. Therefore, in terms of the variables $\partial_{\bar\aleph}\chi$, the motion becomes linear. This identifies the angle variables of this integrable system as the variables conjugate to $\bar z^{\bar\aleph}$ under Legendre transform with respect to the hyperkähler potential χ. To find the most general solution of BPS geodesic motion on $\mathcal{S}$, it only remains to express the complex variables $z^\aleph$ in terms of $\partial_{\bar\aleph}\chi$ and the constants of motion $\bar z^{\bar\aleph}, p_{\bar\aleph}, c_{\bar\aleph}$. Finally, the BPS geodesic motion can be projected on the quaternionic-Kähler base using the covariant c-map formulae of section 3.43, and enforcing the vanishing of the SU(2) momenta.

4.2 Solution

We now exploit the explicit form (3.27) of the hyperkähler potential for the c-map:

$$\chi(\phi^\Lambda, \chi_\Lambda, \tilde\chi_\flat) = 2\sqrt{v^\flat \bar v^\flat}\,\sqrt{\Sigma^2(\phi,\chi) - \tilde\chi_\flat^2}\,, \tag{4.6}$$

where we recall that

$$\phi^\Lambda = \frac{\mathrm{i}}{2}(\xi^\Lambda - \bar\xi^\Lambda)\,, \qquad \chi_\Lambda = \frac{\mathrm{i}}{2}(\tilde\xi_\Lambda - \bar{\tilde\xi}_\Lambda)\,, \tag{4.7}$$

$$\tilde\chi_\flat = w_\flat + \bar w_\flat + \frac{\mathrm{i}}{4}(\xi^\Lambda + \bar\xi^\Lambda)(\tilde\xi_\Lambda - \bar{\tilde\xi}_\Lambda)\,. \tag{4.8}$$

Using the identities

$$\begin{pmatrix} \partial_{\chi_\Lambda}\chi \\ -\partial_{\phi^\Lambda}\chi \end{pmatrix} = \mathrm{Re}\begin{pmatrix} \eta^\Lambda \\ F_\Lambda(\eta) \end{pmatrix} = \begin{pmatrix} x^\Lambda - \frac{x^\flat}{2}\left(\xi^\Lambda + \bar\xi^\Lambda\right) \\ y_\Lambda - \frac{x^\flat}{2}\left(\tilde\xi_\Lambda + \bar{\tilde\xi}_\Lambda\right) \end{pmatrix} \tag{4.9}$$

(where the second equality defines y_Λ) and

$$\partial_{\tilde\chi_\flat}\chi = x^\flat\,, \quad 2\mathrm{i}v^\flat\partial_{v^\flat}\chi = \mathrm{i}\chi - (y_\Lambda - x^\flat\tilde\xi_\Lambda)\,\xi^\Lambda \equiv y_\flat\,, \tag{4.10}$$

where the partial derivatives of χ are taken in the coordinates $(\phi^\Lambda, \chi_\Lambda, \tilde{\chi}_\flat)$, one may rewrite the anti-holomorphic derivatives $\partial_{\bar{z}^{\aleph}}\chi$ appearing in (4.5) as

$$\partial_{\bar{\xi}^\Lambda}\chi = \frac{\mathrm{i}}{2}(y_\Lambda - x^\flat \bar{\tilde{\xi}}_\Lambda) \quad , \qquad -\partial_{\bar{\tilde{\xi}}_\Lambda}\chi = \frac{\mathrm{i}}{2} x^\Lambda \,, \tag{4.11}$$

$$\partial_{\bar{w}_\flat}\chi = x^\flat \quad , \qquad \partial_{\bar{v}^\flat}\chi = \frac{\chi}{2\bar{v}^\flat} \,. \tag{4.12}$$

Together with (4.5), these identities imply that the hyperkähler potential χ is a constant of motion, while $x^\Lambda, y_\Lambda, x^\flat$ flow linearly:

$$x^\Lambda = 2\mathrm{i}\,(P^\Lambda\, t + C^\Lambda)\,, \quad y_\Lambda = 2\mathrm{i}\,(Q_\Lambda\, t + D_\Lambda)\,, \quad x^\flat = 4\mathrm{i}\,(K\, t + E)\,. \tag{4.13}$$

It will be useful to further define

$$\hat{x}^\Lambda \equiv x^\Lambda - x^\flat \bar{\xi}^\Lambda\,, \quad \hat{y}_\Lambda \equiv y_\Lambda - x^\flat \bar{\tilde{\xi}}_\Lambda\,, \tag{4.14}$$

which, like x^Λ and y_Λ, depend linearly on the geodesic time,

$$\hat{x}^\Lambda = \chi(p^\Lambda\, t + c^\Lambda)\,, \quad \hat{y}_\Lambda = \chi(q_\Lambda\, t + d_\Lambda)\,, \quad x^\flat = \chi(k\, t + e)\,, \tag{4.15}$$

with shifted and rescaled momenta,

$$\chi p^\Lambda = 2\mathrm{i}P^\Lambda - 4\mathrm{i}K\bar{\xi}^\Lambda\,, \quad \chi q_\Lambda = 2\mathrm{i}Q_\Lambda - 4\mathrm{i}K\bar{\tilde{\xi}}_\Lambda\,, \quad \chi k = 4\mathrm{i}K\,, \tag{4.16}$$

$$\chi c^\Lambda = 2\mathrm{i}C^\Lambda - 4\mathrm{i}E\bar{\xi}^\Lambda\,, \quad \chi d_\Lambda = 2\mathrm{i}D_\Lambda - 4\mathrm{i}E\bar{\tilde{\xi}}_\Lambda\,, \quad \chi e = 4\mathrm{i}E\,. \tag{4.17}$$

In order to find the explicit trajectory, we note that $(\hat{x}^\Lambda, \hat{y}_\Lambda)$ satisfy "generalized stabilization equations" analogous to (3.17),

$$\frac{1}{2}\left[C\eta_+^\Lambda + \tilde{C}\eta_-^\Lambda\right] = r^\flat\, \hat{x}^\Lambda\,, \quad \frac{1}{2}\left[CF_\Lambda(\eta_+) + \tilde{C}\bar{F}_\Lambda(\eta_-)\right] = r^\flat\, \hat{y}_\Lambda\,, \tag{4.18}$$

where C and $\tilde{C}$ were defined in (3.5). Despite the fact that C and $\tilde{C}$ are in general not complex conjugate to one another, the standard solution (3.19) to the stabilization equations continues to hold,

$$C\eta^\Lambda = r^\flat\left(\hat{x}^\Lambda + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\right), \qquad \tilde{C}\bar{\eta}^\Lambda = r^\flat\left(\hat{x}^\Lambda - \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\right), \tag{4.19a}$$

$$CF_\Lambda = r^\flat\left(\hat{y}_\Lambda - \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\right), \qquad \tilde{C}\bar{F}_\Lambda = r^\flat\left(\hat{y}_\Lambda + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\right). \tag{4.19b}$$

Injecting these relations into (3.16), we may express ϕ^Λ, χ_Λ in terms of the linear flows $\hat{x}^\Lambda, \hat{y}_\Lambda$ as

$$\begin{pmatrix}\phi^\Lambda \\ \chi_\Lambda\end{pmatrix} = \frac{\mathrm{i}x^\flat}{C\tilde{C}}\begin{pmatrix}\hat{x}^\Lambda \\ \hat{y}_\Lambda\end{pmatrix} + \frac{r^\flat}{C\tilde{C}}\begin{pmatrix}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda} \\ -\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\end{pmatrix}. \tag{4.20}$$

Since $(\phi^\Lambda, \chi_\Lambda)$ are related to the differences $\xi^\Lambda - \bar{\xi}^\Lambda$ and $\tilde{\xi}_\Lambda - \bar{\tilde{\xi}}_\Lambda$ by (4.7), and since the anti-holomorphic coordinates $(\bar{\xi}^\Lambda, \bar{\tilde{\xi}}_\Lambda)$ are constants of motion, (4.20) determines $(\xi^\Lambda, \tilde{\xi}_\Lambda)$

once $x^\flat$ and $r^\flat$ are known. The former is given by the linear flow (4.15), while the latter follows from the general property (3.18) of the attractor equations,

$$4(r^\flat)^2\,\Sigma(\hat{x},\hat{y}) = C\tilde{C}\,K(\eta_+,\bar{\eta}_+) = 16\,(r^\flat)^2\,v^\flat\bar{v}^\flat\,\Sigma(\phi,\chi)\,, \tag{4.21}$$

which, in combination with (3.30), leads to

$$r^\flat = \frac{\Sigma(\hat{x},\hat{y})}{\chi}\,. \tag{4.22}$$

Finally, we may obtain the flows of $v^\flat$ and $\tilde{\chi}_\flat$ from (using (3.24))

$$v^\flat = \frac{(r^\flat)^2-(x^\flat)^2}{4\bar{v}^\flat}\,,\quad \tilde{\chi}_\flat = \frac{\chi\,x^\flat}{(r^\flat)^2-(x^\flat)^2}\,, \tag{4.23}$$

and then infer the flow of $w_\flat$ from (4.8). We have now obtained all of the holomorphic coordinates $z^\aleph$, and hence determined the BPS geodesic trajectory on the hyperkähler cone $\mathcal{S}$.

In order to project the geodesic flow to the quaternionic-Kähler base we use the covariant c-map formulae of section 3.4. In view of (3.38), the evolution of the scalars $(z^a,\bar{z}^{\bar{a}})$ is simply given by the ratios

$$z^\Lambda = \frac{\hat{x}^\Lambda + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}}{\hat{x}^0 + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_0}}\,. \tag{4.24}$$

The evolution of the dilatonic variable U follows from (3.34) and (4.22),

$$\mathrm{e}^{-2U} = \frac{4\,\Sigma(\hat{x},\hat{y})}{\chi^2}\,. \tag{4.25}$$

In the black hole context, the time t is the inverse radial distance in the spatial slices of the black hole, while the area of the sphere as a function of the radial distance is given by $4\pi\mathrm{e}^{-2U}/t^2$. Using (4.15), the Bekenstein-Hawking entropy of the black hole is therefore given by

$$\boxed{S_{\mathrm{BH}}(p,q) = \lim_{t\to\infty}\pi\mathrm{e}^{-2U}/t^2 = 4\pi\,\Sigma(p^\Lambda,q_\Lambda)} \tag{4.26}$$

reproducing the known relation between black hole entropy and Hesse potential [8, 41].

The motion of $\zeta^\Lambda,\tilde{\zeta}_\Lambda$ may be obtained by rewriting (3.45), (3.50) in terms of $C,\tilde{C}$ and substituting (4.19),

$$\zeta^\Lambda = \bar{\xi}^\Lambda - \frac{1}{2(r^\flat)^2}\left[C\eta_+^\Lambda - \tilde{C}\bar{\eta}_+^\Lambda\right] = \bar{\xi}^\Lambda - \frac{\mathrm{i}}{r^\flat}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\,, \tag{4.27a}$$

$$\tilde{\zeta}_\Lambda = \bar{\tilde{\xi}}_\Lambda - \frac{1}{2(r^\flat)^2}\left[CF_\Lambda - \tilde{C}\bar{F}_\Lambda\right] = \bar{\tilde{\xi}}_\Lambda + \frac{\mathrm{i}}{r^\flat}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\,. \tag{4.27b}$$

Finally, the flow of σ follows from the complex conjugate of (3.52),

$$\sigma = \bar{\alpha} + \frac{\mathrm{i}}{r^\flat}\left[\bar{\xi}^\Lambda\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda} + \bar{\tilde{\xi}}^\Lambda\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\right]\,. \tag{4.28}$$

It is also useful to note the solution for the holomorphic coordinates $\xi^\Lambda, \tilde{\xi}_\Lambda$, obtained by similar manipulations:

$$\xi^\Lambda = \zeta^\Lambda + \frac{1}{2r^\flat}\left[\frac{C}{\tilde{C}}\left(\hat{x}^\Lambda - \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\right) - \frac{\tilde{C}}{C}\left(\hat{x}^\Lambda + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{y}_\Lambda}\right)\right], \tag{4.29a}$$

$$\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + \frac{1}{2r^\flat}\left[\frac{C}{\tilde{C}}\left(\hat{y}_\Lambda + \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\right) - \frac{\tilde{C}}{C}\left(\hat{y}_\Lambda - \mathrm{i}\frac{\partial\Sigma(\hat{x},\hat{y})}{\partial\hat{x}^\Lambda}\right)\right]. \tag{4.29b}$$

We recall that geodesic motion on $\mathcal{S}$ only projects down to geodesic motion on the base $\mathcal{M}$ when the SU(2) momentum vanishes.[9] The $\mathrm{U}(1) \subset \mathrm{SU}(2)$ charge has already been set to zero in (4.3d), and J_+ vanishes for all holomorphic geodesics, but it remains to enforce

$$J_- = \hat{x}^\Lambda p_{\bar{\xi}^\Lambda} - \hat{y}_\Lambda p_{\bar{\tilde{\xi}}_\Lambda} - \frac{\mathrm{i}}{4}\bar{y}_\flat p_{\bar{w}_\flat} + x^\flat p_{\bar{v}^\flat} = 0\,. \tag{4.30}$$

This determines the NUT charge k as

$$\mathrm{i}k = p^\Lambda d_\Lambda - q_\Lambda c^\Lambda\,. \tag{4.31}$$

Thus, we have obtained the general BPS trajectory on the complexification of the quaternionic-Kähler metric (2.41).

It remains to enforce the reality conditions appropriate to the problem at hand. For BPS instantons, there is no such reality condition, although one should in principle ensure that the Euclidean configuration can be reached by analytic continuation of the path integral. For BPS black holes, (U,σ) need to be real whereas $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ need to be imaginary. This requires the charges (P^Λ, Q_Λ) to be imaginary and K real, while the situation is reversed for $(p^\Lambda, q_\Lambda, k)$. Thus, (x^Λ, y_Λ) need to be real while $x^\flat$ is imaginary and $r^\flat$ is real. Moreover, one should demand that $(\bar{\xi}^\Lambda, \bar{\tilde{\xi}}_\Lambda)$ are imaginary and $\bar{\alpha}$ is real. One may check from (4.29), (3.52) that this requires that $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ are imaginary while α is real.

We conclude that the twistor space for the para-quaternionic-Kähler space $\mathcal{M}^*$ is obtained by taking $(\xi^\Lambda, \tilde{\xi}_\Lambda, \bar{\xi}^\Lambda, \bar{\tilde{\xi}}_\Lambda)$ as independent, purely imaginary variables, and $(\alpha, \bar{\alpha})$ as independent, real variables. We note that then $z\bar{z}$ becomes a phase, as a consequence of (3.48), and the S^2 fiber of the twistor space becomes a hyperbolic two-plane H_2.

We may now compare our solution to the ones appearing in [45 − 49, 15]. Using (3.46), we may express $(\eta_+^\Lambda, F_\Lambda(\eta_+))$ in terms of the symplectic section $(X^\Lambda, F_\Lambda(X))$ on the special Kähler manifold $\mathcal{M}_s$, and obtain

$$-\mathrm{i}\,\bar{z}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} + \mathrm{i}\,\bar{z}^{-1}\,\mathrm{e}^{U+\frac{1}{2}\mathcal{K}(X,\bar{X})}\begin{pmatrix} \bar{X}^\Lambda \\ \bar{F}_\Lambda \end{pmatrix} = \frac{1}{r^\flat}\begin{pmatrix} \hat{x}^\Lambda \\ \hat{y}_\Lambda \end{pmatrix}. \tag{4.32}$$

Further expressing $r^\flat$ in terms of the dilaton using (3.34), and setting $\bar{z} = 1/(\bar{z})^* = \mathrm{e}^{-\mathrm{i}\alpha}$, we find

$$\mathrm{Im}\left[\mathrm{e}^{\frac{1}{2}\mathcal{K}(X,\bar{X})-U-\mathrm{i}\alpha}\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}\right] = \begin{pmatrix} H^\Lambda \\ H_\Lambda \end{pmatrix} \tag{4.33}$$

[9]More general geodesic motion on $\mathcal{S}$ would descend to motion on $\mathcal{M}$ in a non-trivial magnetic field.

with

$$H^{\Lambda} = \hat{x}^{\Lambda}/\chi = p^{\Lambda} t + c^{\Lambda} , \quad H_{\Lambda} = \hat{y}_{\Lambda}/\chi = q_{\Lambda} t + d_{\Lambda} . \tag{4.34}$$

In the black hole problem, t is identified with the inverse radial distance on the $\mathbb{R}^3$ spatial slices, so $(H^{\Lambda}, H_{\Lambda})$ are indeed harmonic functions, although not of the most general type allowed in [45 − 48]. In particular, we see that the phase α (equal to the phase of the central charge Z near the horizon) is identified throughout the flow as the azimuthal angle on the S^2 fiber of $\mathcal{Z}$. It would be very interesting to generalize our discussion to the multi-centered case, and lift the standard solutions to suitably holomorphic maps from $\mathbb{R}^3$ to $\mathcal{S}$.

5. The quaternionic Penrose transform

The classic Penrose transform relates wave functions on subsets of the twistor space $\mathbb{CP}^3$ — more precisely, elements in the sheaf cohomology $H^1(X \subset \mathbb{CP}^3, \mathcal{O}(-2h-2))$ — to helicity-h solutions of conformally invariant wave equations on subsets of $\mathbb{R}^4$ (see e.g. [50 − 52]). This transform has been generalized in many directions. For example, one may replace $\mathbb{R}^4$ by another self-dual 4-manifold [53, 54]. Self-dual 4-manifolds can be considered as the $n = 1$ case of quaternionic-Kähler $4n$-manifolds, and there is a further extension of the Penrose transform to this case [21, 22]. Letting $\mathcal{M}$ be a quaternionic-Kähler manifold and $\mathcal{Z}$ its twistor space, this "quaternionic Penrose transform" relates elements in $H^1(Y \subset \mathcal{Z}, \mathcal{O}(-k))$ to solutions of wave equations constructed from the quaternionic structure on subsets of $\mathcal{M}$.

Here we work out one aspect of this transform explicitly: we represent elements $\psi_f \in H^1(Y \subset \mathcal{Z}, \mathcal{O}(-k))$ by holomorphic sections f on an appropriate open set in $\mathcal{Z}$, and we give a contour integral formula which transforms any such f into a solution of a wave equation on $\mathcal{M}$. Furthermore, in the case where $\mathcal{M}$ comes from the c-map, we use the results of section 3.5 to make this transform particularly concrete: it converts holomorphic functions in $2n + 1$ variables, $g(\xi^{\Lambda}, \tilde{\xi}_{\Lambda}, \alpha)$, to solutions of wave equations on $\mathcal{M}$. Finally we use this formalism to compute the Penrose transform of particular eigenstates of the Heisenberg group; physically, these are interpreted as the wave function of BPS black holes in radial quantization [17].

5.1 For general quaternionic-Kähler manifolds

Suppose $\mathcal{M}$ is a quaternionic-Kähler manifold. As we reviewed in section 2.1, there is a natural $\mathbb{CP}^1$-bundle over $\mathcal{M}$, the twistor space $\mathcal{Z}$, equipped with a canonical complex structure. The Penrose transform is local on $\mathcal{M}$, so we may as well take $\mathcal{M}$ to be a small open set; in particular, we may assume that the decomposition $T_{\mathbb{C}}\mathcal{M} = E \otimes H$ exists globally on $\mathcal{M}$. Then $\mathcal{Z}$ is equipped with a canonical holomorphic line bundle $\mathcal{O}(1)$. We choose a standard local trivialization of H, and let (π^1, π^2) be the corresponding coordinates on the total space of $H^{\times}$. Then $z = \pi^1/\pi^2$ is a coordinate on the $\mathbb{CP}^1$ fibers of $\mathcal{Z}$.

To construct the Penrose transform we begin by constructing a class ψ_f in the sheaf cohomology $H^1(\mathcal{Z}, \mathcal{O}(-k))$. The cohomological interpretation of the Penrose transform has been described in [55, 51]; we construct ψ_f using the Čech description of the cohomology, reviewed e.g. in [56, 51]. The Čech construction depends on a covering of $\mathcal{Z}$ by open sets: we take two sets, $U_1 = \{(x,z) : x \in \mathcal{M}, z \neq 0\}$ and $U_2 = \{(x,z) : x \in \mathcal{M}, z \neq \infty\}$. Then ψ_f is represented simply by a holomorphic section f of $\mathcal{O}(-k)$ on $U_1 \cap U_2$. Equivalently, we may regard f as a function $f(x, \pi^1, \pi^2)$, homogeneous of degree $-k$ in the $\pi^{A'}$, defined where $\pi^1 \neq 0$, $\pi^2 \neq 0$, and holomorphic. Here "holomorphic" is defined using the complex structure on the total space of $\mathcal{O}(-1) \to \mathcal{Z}$, described in section 2: concretely, it implies that all the vector fields

$$d_A \equiv \pi^{A'} \partial_{AA'} - \pi^{B'} \pi^{C'} (p_{AB'})^{D'}_{C'} \frac{\partial}{\partial \pi^{D'}} \tag{5.1}$$

annihilate f. (To check this, one shows that each d_A has zero inner product with the basis of $(1,0)$-forms on $\mathcal{S}$ described in section 2, so it is a $(0,1)$ vector field, i.e. an antiholomorphic derivative.)

We now construct the Penrose transform φ of ψ_f as an appropriate contour integral of f. For $k > 2$, we will show by simple manipulations that the holomorphy of f implies φ obeys first-order differential equations on $\mathcal{M}$. We then turn to the $k = 2$ case, which leads to second-order differential equations and is technically more difficult. The Penrose transform for $k < 2$ will involve in general differentiation as well as integration, but should be treatable along the same lines as in the classic case; we do not consider it here.

For $\mathcal{O}(-k)$, $k > 2$: for notational simplicity we treat mainly the case $k = 3$. So given f representing $\psi_f \in H^1(\mathcal{Z}, \mathcal{O}(-3))$, we construct a field on $\mathcal{M}$ by

$$\boxed{\varphi^{A'}(x) = \oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) \pi^{A'} f(x,\pi)} \tag{5.2}$$

Since f has homogeneity -3 and there are 3 explicit factors of π, the whole integrand has homogeneity 0, so it is well defined on $\mathcal{Z}$. The contour of integration is chosen to lie in the $\mathbb{CP}^1$ fiber of $\mathcal{Z}$ over x, and to separate $z = 0$ and $z = \infty$.

In the rest of this section, we will prove that $\varphi^{A'}(x)$ obeys a Dirac-type equation,

$$\nabla_{AA'} \varphi^{A'}(x) = 0\,. \tag{5.3}$$

The strategy is simple: because f is holomorphic on $\mathcal{Z}$ we have $d_A f = 0$. We insert this into the contour integral (5.2) to get

$$0 = \oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) \left(\pi^{A'} \partial_{AA'} - \pi^{E'} \pi^{A'} (p_{AE'})^{G'}_{A'} \frac{\partial}{\partial \pi^{G'}}\right) f(x,\pi)\,. \tag{5.4}$$

Now integrate the operator $\frac{\partial}{\partial \pi^{G'}}$ by parts, using the identity (easily checked in local coordinates)

$$\oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) \frac{\partial}{\partial \pi^{G'}} g(\pi) = 0\,. \tag{5.5}$$

Applied to (5.4) this integration by parts gives two terms, since $\frac{\partial}{\partial \pi^{G'}}$ can hit either $\pi^{E'}$ or $\pi^{A'}$. If it hits A' we get $(p_{AE'})^{A'}_{A'}$ which vanishes; so we only get the E' term, giving

$$0 = \oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) \pi^{A'} \left(\partial_{AA'} + (p_{AE'})^{E'}_{A'}\right) f(x,\pi)\,. \tag{5.6}$$

The right side is $\nabla_{AA'}\varphi^{A'}$, so we get the desired (5.3).

More generally, the twistor transform for $\mathcal{O}(-k)$ with $k > 2$ gives totally symmetric $(k-2)$-tensors on $\mathcal{M}$; it is obtained by replacing (5.2) with

$$\varphi^{(A'_1 A'_2 \cdots A'_{k-2})}(x) = \oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) \pi^{A'_1}\pi^{A'_2} \cdots \pi^{A'_{k-2}} f(x,\pi)\,, \tag{5.7}$$

and the same differentiation under the integral sign we did above shows

$$\boxed{\nabla_{AA'_1}\, \varphi^{(A'_1 A'_2 \cdots A'_{k-2})}(x) = 0} \tag{5.8}$$

For $\mathcal{O}(-2)$: now we turn to the harder case $k = 2$. Given f representing $\psi_f \in H^1(\mathcal{Z}, \mathcal{O}(-2))$, we construct a scalar function on $\mathcal{M}$ by a contour integral similar to (5.2),

$$\boxed{\varphi(x) = \oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) f(x,\pi)} \tag{5.9}$$

In the rest of this section we prove that $\varphi(x)$ obeys a family of second-order differential equations,

$$\left(\nabla_{AA'}\nabla^{A'}_{B} - \nu\epsilon_{AB}\right)\varphi(x) = 0\,, \tag{5.10}$$

where we recall that $\nu = \frac{1}{4n(n+2)}R$. Quaternionic geometry in case $n = 1$ reduces to conformal geometry, and correspondingly (5.10) reduces to the conformal Laplacian $\Delta - \frac{1}{6}R$ in that case. For $n > 1$, (5.10) gives more than one equation, transforming in $\wedge^2(E)$; tracing them with ϵ^{AB} gives $\Delta - \frac{1}{2(n+2)}R$, which differs from the conformal Laplacian $\Delta - \frac{4n-2}{4(4n-1)}R$.

To establish (5.10) we begin by defining a second differential operator d'_B on $\mathcal{S}$, acting on sections $f_A(x,\pi)$ by

$$d'_B f_A = \left(\pi^{A'}\partial_{BA'} - \pi^{B'}\pi^{C'}(p_{BB'})^{D'}_{C'}\frac{\partial}{\partial \pi^{D'}}\right) f_A + \pi^{A'}(q_{BA'})^{D}_{A} f_D\,. \tag{5.11}$$

This operator is engineered to obey

$$d'_A d_B - d'_B d_A = 0\,. \tag{5.12}$$

To prove this, we choose a local frame such that $p = 0$ at x: then (5.12) is

$$\pi^{A'}\pi^{B'}\pi^{C'}\left(\partial_{AA'}(p_{BB'})^{D'}_{C'} - \partial_{BA'}(p_{AB'})^{D'}_{C'}\right)\frac{\partial}{\partial\pi^{D'}} = 0\,. \tag{5.13}$$

On the other hand, in these coordinates the formula (2.6) for the $USp(2)$ curvature becomes

$$\partial_{AA'}(p_{BB'})^{D'}_{C'} - \partial_{BB'}(p_{AA'})^{D'}_{C'} = \frac{\nu}{2}\epsilon_{AB}(\delta^{D'}_{A'}\epsilon_{C'B'} + \delta^{D'}_{B'}\epsilon_{C'A'})\,. \tag{5.14}$$

This vanishes when symmetrized over $(A'B'C')$, establishing (5.13).

As we will now see, the complex

$$\mathcal{O}(-2) \stackrel{d_A}{\to} \mathcal{O}_A(-1) \stackrel{d'_B}{\to} \mathcal{O}_{AB}(0) \tag{5.15}$$

on $\mathcal{Z}$ leads to a differential equation on $\mathcal{M}$, which will turn out to be (5.10). Abstractly this equation arises as one of the differentials in a spectral sequence computing $H^*(\mathcal{Z}, \mathcal{O}(-2))$, as sketched in [22], along the lines of similar arguments described in [55, 57]; but in this case the construction is simple enough that it can be worked out by hand. We begin by considering the function

$$h(x,\pi) \equiv \frac{\varphi(x)}{\pi^1\pi^2} - f(x,\pi) \tag{5.16}$$

on $\mathcal{S}$. By construction, contour-integrating h gives

$$\oint \left(\pi_{B'} \mathrm{d}\pi^{B'}\right) h(x,\pi) = 0\,. \tag{5.17}$$

But since $H^1(\mathbb{CP}^1, \mathcal{O}(-2))$ is one-dimensional, this vanishing implies that $h(x,\pi)$ is trivial in $H^1(\mathbb{CP}^1, \mathcal{O}(-2))$; in other words, there is a decomposition $h = h^{(1)} + h^{(2)}$ where $h^{(i)}$ is defined on U_i. Applying d_A to this,

$$d_A h = d_A h^{(1)} + d_A h^{(2)}\,. \tag{5.18}$$

Moreover, this is the unique decomposition of $d_A h$ into a piece defined on U_1 and a piece defined on U_2: the uniqueness follows from the fact that the ambiguity would be a global section of $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, and there are no such sections. We now compute this decomposition in another way: using $d_A f = 0$ we see that $d_A h = d_A\left(\frac{\varphi}{\pi^1\pi^2}\right)$, and using the definition (5.1) of d_A this gives

$$d_A h = \frac{1}{\pi^1}\left[\partial_{A2} + (p_{A1})^1_2 + \frac{\pi^2}{\pi^1}(p_{A2})^1_2\right]\varphi + \frac{1}{\pi^2}\left[\partial_{A1} + (p_{A2})^2_1 + \frac{\pi^1}{\pi^2}(p_{A1})^2_1\right]\varphi\,. \tag{5.19}$$

The first (resp. second) term is defined on U_1 (resp. U_2), so by the uniqueness of the decomposition,

$$d_A h^{(1)} = \frac{1}{\pi^1}\left[\partial_{A2} + (p_{A1})^1_2 + \frac{\pi^2}{\pi^1}(p_{A2})^1_2\right]\varphi\,. \tag{5.20}$$

Then using (5.12),

$$(d'_A d_B - d'_B d_A) h^{(1)} = 0\,, \tag{5.21}$$

gives a second-order differential equation for φ. To work out this equation it is again convenient to work in normal coordinates on $\mathcal{M}$, with $p = q = 0$ at x. Then substituting the definition (5.11) of d'_B and (5.20) in (5.21), the terms proportional to π^2/π^1 vanish using (5.14), leaving

$$\left(\partial_{A1}\partial_{B2} - \partial_{B1}\partial_{A2} + \partial_{A1}(p_{B1})^1_2 - \partial_{B1}(p_{A1})^1_2\right)\varphi(x) = 0\,. \tag{5.22}$$

This is not written in a manifestly $USp(2)$-covariant way, which can be traced back to the fact that our covering of $\mathcal{Z}$ by $\{U_1, U_2\}$ is not covariant. Using (5.14), it is easily rewritten in the desired form,

$$\boxed{\left(\nabla_{AA'}\nabla^{A'}_B - \nu\,\epsilon_{AB}\right)\varphi(x) = 0} \tag{5.23}$$

5.2 For c-map spaces

This general construction can be made more explicit when the quaternionic-Kähler space $\mathcal{M}$ arises from the c-map: as we will see, in this case the Penrose transform allows us to construct solutions of wave equations on $\mathcal{M}$ starting from arbitrary holomorphic functions in $2n+1$ variables $g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$.

So suppose $\mathcal{M}$ comes from the c-map. Then we have local complex coordinates $(\xi, \tilde{\xi}, \alpha)$ which cover an open set in $\mathcal{Z}$. To be precise, using (3.55), we see that this coordinate system covers all of each twistor sphere except the north and south poles. We also have a natural trivialization of $\mathcal{O}(-k)$, provided by the $\mathbb{C}^\times$ gauge condition $v^\flat = 1$, which enables us to pass between homogeneous functions on $\mathcal{S}$ and ordinary functions of $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$. So in this gauge a class in $H^1(\mathcal{Z}, \mathcal{O}(-k))$ can be simply represented by a holomorphic function $g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$, while the integration measure in (5.9) is obtained by using (3.56) in the $v^\flat = 1$ gauge. Thus, the Penrose transform for scalar fields (5.9) becomes

$$\varphi(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 4\ \mathrm{e}^{2U} \oint \frac{\mathrm{d}z}{z}\, g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)\,, \tag{5.24}$$

where ξ^Λ, $\tilde{\xi}_\Lambda$, α are given in terms of $(U, X, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$ by (3.55). Similarly (5.7) becomes

$$\boxed{\varphi^{(A'_1 A'_2 \cdots A'_{k-2})}(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 2^k\ \mathrm{e}^{kU} \oint \tfrac{\mathrm{d}z}{z}\, z^{\frac{1}{2}\delta} g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)} \tag{5.25}$$

where $\delta \equiv$ ((the number of i with $A'_i = 1$) $-$ (the number of i with $A'_i = 2$)).

When k is odd, (5.25) involves square roots of z. This is related to the fact that $\mathcal{S}$ provides a global definition of $\mathcal{O}(k)$ and $S^k(H)$ only for k even. It is not obvious whether one can make sense of (5.25) for k odd (perhaps by choosing g with some appropriate branch cuts).

5.3 The inner product

Since we view $\psi \in H^1(\mathcal{Z}, \mathcal{O}(-k))$ as a wave function, it is natural to ask whether there is a canonical inner product $\langle\psi|\psi'\rangle$ defined in terms of the geometry of $\mathcal{Z}$. For the classical case where $\mathcal{Z}$ is a subset of $\mathbb{CP}^3$, the answer to this question can be phrased in terms of an isomorphism $H^1(\mathcal{Z}, \mathcal{O}(-k)) \simeq H^1(\mathcal{Z}, \mathcal{O}(k-4))$ known as the "twistor transform" [58]. Upon representing the classes in H^1 by holomorphic functions, the corresponding inner product admits a concrete integral representation, given e.g. in section 3.3 of [59]. While we do not know the generalization of the twistor transform to the c-map case, we can construct a candidate for an inner product,

$$\langle\psi_f|\psi_{f'}\rangle = \int_{\mathcal{Z}'} \mathrm{vol}_{\mathcal{Z}}\, \langle f|f'\rangle \tag{5.26}$$

Here, $\langle f|f'\rangle$ is the Hermitian inner product in $\mathcal{O}(-k)$ and $\mathrm{vol}_{\mathcal{Z}}$ the volume form induced from the Kähler -Einstein metric on $\mathcal{Z}$. This formula is not well defined *a priori*; it only makes sense under an assumption about the global structure of $\mathcal{Z}$, namely, every ψ is obtained as ψ_f, with f a unique holomorphic section of $\mathcal{O}(-k)$ over $\mathcal{Z}' = \{z \neq 0, z \neq$

$\infty\} \subset \mathcal{Z}$. There is some evidence that this assumption does hold when $\mathcal{M}$ is a symmetric space [26].

Choosing the $v^\flat = 1$ gauge to trivialize $\mathcal{O}(-k)$, f and f' become $g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ and $g'(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ as above, so

$$\langle f|f'\rangle = \bar{g}\, g'\, \mathrm{e}^{kK_{\mathcal{Z}}} \tag{5.27}$$

where $K_{\mathcal{Z}}$ is given in (3.32). We determine the volume form by considering the line bundle K of holomorphic top-forms on $\mathcal{Z}$. K admits a natural Hermitian metric, in which the squared norm of any ω is $\frac{\omega\wedge\bar{\omega}}{\mathrm{vol}_{\mathcal{Z}}}$. On the other hand $K \simeq \mathcal{O}(-2n-2)$, and the metric is the $(2n+2)$-th power of the metric in $\mathcal{O}(-1)$ [18]. Recall from section 2 that the squared norm of the Heisenberg invariant section $v^\flat = 1$ is by definition $\mathrm{e}^{K_{\mathcal{Z}}}$; hence the Heisenberg invariant section $\mathrm{d}\xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \mathrm{d}\alpha$ has squared norm $\mathrm{e}^{(2n+2)K_{\mathcal{Z}}}$, up to an overall constant. Comparing these two gives

$$\mathrm{vol}_{\mathcal{Z}} = \mathrm{e}^{-(2n+2)K_{\mathcal{Z}}}\, \mathrm{d}\xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \mathrm{d}\alpha\, \mathrm{d}\bar{\xi}^\Lambda \mathrm{d}\bar{\tilde{\xi}}_\Lambda \mathrm{d}\bar{\alpha}\,. \tag{5.28}$$

So altogether we find the inner product of wave functions as

$$\boxed{\langle\psi_f|\psi_{f'}\rangle = \int_{\mathcal{Z}'} \mathrm{d}\xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \mathrm{d}\alpha\, \mathrm{d}\bar{\xi}^\Lambda \mathrm{d}\bar{\tilde{\xi}}_\Lambda \mathrm{d}\bar{\alpha}\ \mathrm{e}^{(k-2n-2)K_{\mathcal{Z}}}\, \overline{g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)}\, g'(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)} \tag{5.29}$$

$\mathcal{Z}'$ does not cover the full range of the complex coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$: the integration should run over a domain such that the bracket in (3.32) is strictly positive.

5.4 The BPS black hole wave function

As an example of this technology, we compute the Penrose transform of a class in $H^1(\mathcal{Z}, \mathcal{O}(-k))$ $(k \geq 2)$ which is an eigenvector for the Heisenberg group acting on $\mathcal{Z}$, with vanishing central character. As will be explained in [17], such a class describes the wave function of a BPS black hole with fixed real electric and magnetic charges (q_Λ, p^Λ) and vanishing NUT charge, in a mini-superspace radial quantization scheme.

Given the action (3.33) of the complexified Heisenberg algebra on the twistor space $\mathcal{Z}$, an eigenvector is determined up to normalization by its eigenvalues $\mathrm{i}p^\Lambda, \mathrm{i}q_\Lambda$ under P^Λ and Q_Λ,

$$g(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) = \mathrm{e}^{\mathrm{i}(p^\Lambda\tilde{\xi}_\Lambda - q_\Lambda\xi^\Lambda)}\,. \tag{5.30}$$

We expect that this wave function is delta-function normalizable with respect to the inner product (5.29) (perhaps after regulating by appropriately continuing k.) In physical applications, one would expect to consider a quotient of $\mathcal{Z}$ by a lattice in the Heisenberg group, which would select integer momenta p^I, q_I; the wave function (5.30) then should become normalizable, as the flat directions $\xi^\Lambda - \bar{\xi}^\Lambda$, $\tilde{\xi}^\Lambda - \bar{\tilde{\xi}}^\Lambda$, $\alpha - \bar{\alpha}$ become compact.

We now compute the Penrose transform of (5.30), starting with the case $k = 2$. Equation (5.24) gives

$$\varphi(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 4\mathrm{e}^{2U + \mathrm{i}p^\Lambda\tilde{\zeta}_\Lambda - \mathrm{i}q_\Lambda\zeta^\Lambda} \oint \frac{\mathrm{d}z}{z} \exp\left(-\mathrm{e}^U (z\bar{Z} + z^{-1}Z)\right), \tag{5.31}$$

where

$$Z = \mathrm{e}^{\frac{1}{2}\mathcal{K}(X,\bar{X})}(p^\Lambda F_\Lambda(X) - q_\Lambda X^\Lambda) \tag{5.32}$$

is the "central charge", familiar from $\mathcal{N} = 2$ supergravity. Using $\oint \frac{\mathrm{d}z}{z} \mathrm{e}^{az+bz^{-1}} = 2\pi I_0(2\sqrt{ab})$ (the modified Bessel function),

$$\varphi(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 8\pi\, \mathrm{e}^{2U + \mathrm{i}p^\Lambda \tilde{\zeta}_\Lambda - \mathrm{i}q_\Lambda \zeta^\Lambda}\; I_0(2\mathrm{e}^U |Z|) \tag{5.33}$$

More generally for $k \geq 2$, (5.24) or (5.25) give

$$\varphi^{(A'_1 A'_2 \cdots A'_{k-2})}(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 2^k\, \mathrm{e}^{kU + \mathrm{i}p^\Lambda \tilde{\zeta}_\Lambda - \mathrm{i}q_\Lambda \zeta^\Lambda} \oint \frac{\mathrm{d}z}{z} z^{\frac{1}{2}\delta} \exp\left(-\mathrm{e}^U (z\bar{Z} + z^{-1} Z)\right), \tag{5.34}$$

and using $\oint \frac{\mathrm{d}z}{z} z^m \mathrm{e}^{az+bz^{-1}} = 2\pi \left(\frac{a}{b}\right)^{\frac{m}{2}} I_{-m}(2\sqrt{ab})$,

$$\boxed{\varphi^{(A'_1 A'_2 \cdots A'_{k-2})}(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = 2^{k+1}\pi\, \mathrm{e}^{kU + \mathrm{i}p^\Lambda \tilde{\zeta}_\Lambda - \mathrm{i}q_\Lambda \zeta^\Lambda} \left(\frac{\bar{Z}}{Z}\right)^{\frac{\delta}{4}} I_{-\frac{\delta}{2}}(2\mathrm{e}^U |Z|)} \tag{5.35}$$

at least when k (and hence δ) is even. Irrespective of the value of k, we see that in the "weak coupling" or "near horizon" limit $U \to -\infty$, the wave function φ as a function on the special Kähler manifold has minima at the minima of $|Z|$, and grows exponentially away from these points. In the application to black hole physics, however, the required analytic continuation of the charges turns the modified Bessel function I into a J Bessel function, with phase stationary at the stationary points of the central charge $|Z|$, and modulus power-suppressed away from these points. This is consistent with the classical attractor behavior [37 – 39], although the absence of exponential decay is perhaps unexpected. We shall return to the physical interpretation of this wave function in the black hole context in [17], and in the instanton context in [60].

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A. Details on the superconformal quotient

In this appendix, we give some more details and useful formulas to prove the results discussed in section 3.4. The superconformal quotient can be performed either on the tensor multiplet side or on the hypermultiplet side: we choose the former. In the notation of [43] (but with $\mathcal{K} \to -\mathcal{K}$) the relevant bosonic terms of the tensor multiplet Lagrangian coupled to Poincaré supergravity after the c-map read

$$\begin{aligned} e^{-1} L = &-\frac{1}{2}(\partial_\mu \phi)^2 - 2\,\mathcal{G}_{a\bar{b}} \partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \frac{1}{2} \mathrm{e}^{-\phi} (\mathcal{N} + \bar{\mathcal{N}})_{\Lambda\Sigma} \partial_\mu A^\Lambda \partial^\mu A^\Sigma \\ &+ 2\,\mathcal{T}_{IJ}\, E^I_\mu E^{J\mu} + \mathrm{i}(\mathcal{N} - \bar{\mathcal{N}})_{\Lambda\Sigma} \left[(\partial_\mu A^\Lambda) E^{\Sigma\mu} - 2(\partial_\mu A^\Lambda) A^\Sigma E^{0\mu}\right]. \end{aligned} \tag{A.1}$$

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Here $E^\mu = \frac{\mathrm{i}}{2} e^{-1}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu E_{\rho\sigma}$ is the field strength of the antisymmetric tensor field $E_{\mu\nu}$. The index I runs over one more value than Λ, so $I = \{\flat, \Lambda\}$. The matrix $\mathcal{T}_{IJ}$ appearing in the tensor field kinetic term is given by

$$\mathcal{T}_{IJ} = \mathrm{e}^\phi \begin{bmatrix} \mathrm{e}^\phi - (\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma}A^\Lambda A^\Sigma & \frac{1}{2}(\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma}A^\Lambda \\ \frac{1}{2}(\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma}A^\Sigma & -\frac{1}{4}(\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma} \end{bmatrix}, \tag{A.2}$$

where $\mathcal{N}$ is defined as in (2.42). The relation between these variables and those of the main text is given in footnote 3 on page 13.

Our task is to prove that the Lagrangian (A.1) follows from the superspace Lagrangian density $\mathcal{L}$ given in (3.9). The component Lagrangian for the rigid superconformal supersymmetric tensor fields follows from [33, 20]. The relevant terms of the bosonic Lagrangian read

$$\begin{aligned} L = {} & \mathcal{L}_{x^I x^J}\Big(-\frac{1}{4}\partial_\mu x^I \partial^\mu x^J - \partial_\mu v^I \partial^\mu \bar{v}^J + E^I_\mu E^{\mu J}\Big) \\ & -\mathrm{i}E^I_\mu \Big(\mathcal{L}_{v^I x^J}\partial^\mu v^J - \mathcal{L}_{\bar{v}^I x^J}\partial^\mu \bar{v}^J\Big). \end{aligned} \tag{A.3}$$

The Lagrangian (A.3) has conformal symmetry. The scaling weight of the scalars is 2 and the matrix of second derivatives of $\mathcal{L}$ has scaling weight minus two. The function $\mathcal{L}$ itself has therefore scaling weight plus two. From now on, we denote $\mathcal{L}_{IJ} \equiv \mathcal{L}_{x^I x^J}$. To obtain the Poincaré theory for the tensor multiplets, one first couples to the Weyl multiplet and integrates out the SU(2) gauge fields. This procedure is called the superconformal quotient and was carried out in [61, 43]. We will here apply it to the case of the c-map. It suffices to consider only the terms quadratic in the tensor fields, the rest is fixed by supersymmetry. Following [61, 43], the zero weight matrix that multiplies the two tensors in the Poincaré theory is

$$e^{-1}L_{\mathrm{TT}} = \mathcal{H}_{IJ}E^I_\mu E^{\mu J}\,, \tag{A.4}$$

with

$$\mathcal{H}_{IJ} = \chi_{\mathrm{T}}\Big(\mathcal{L}_{IJ} + \frac{1}{4}\mathcal{L}_{IK}(\vec{r}^{\,K}\cdot M^{-1}\vec{r}^{\,L})\mathcal{L}_{LJ}\Big)\,, \tag{A.5}$$

where the tensor potential is defined as

$$\chi_{\mathrm{T}}(v,\bar{v},x) \equiv -\mathcal{L} + x^I \mathcal{L}_I\,, \tag{A.6}$$

and the matrix M appearing in the inner product is defined as

$$M^{rs} = \frac{1}{4}\Big[\mathcal{L}_{IJ}(\vec{r}^{\,I}\cdot\vec{r}^{\,J})\delta^{rs} - r^{Ir}\mathcal{L}_{IJ}r^{Js}\Big], \tag{A.7}$$

with the indices r,s running over three values, and $\vec{r}$ as in (2.34).

Notice that the tensor potential χ_T is related to the hyperkähler potential χ as defined in (2.30). In fact, after eliminating the scalars x^I in terms of $w_I + \bar{w}_I$ by the Legendre transform, they become the same function, up to a sign,

$$\chi_{\mathrm{T}}\Big(v,\bar{v},x(w+\bar{w},v,\bar{v})\Big) = -\chi(v,\bar{v},w+\bar{w})\,. \tag{A.8}$$

Using the results of the main text (3.31), we have

$$\chi_{\mathrm{T}}(v,\bar{v},x) = -\frac{v^\flat \bar{v}^\flat}{(r^\flat)^3} K(\eta_+,\eta_-)\,. \tag{A.9}$$

To make further progress, we list some of the second derivatives of $\mathcal{L}$. To facilitate the notation we introduce the scale and SU(2) invariant variables

$$A^\Lambda = \frac{1}{2(r^\flat)^2}\Big(x^\flat x^\Lambda + 2(v^\flat \bar{v}^\Lambda + \bar{v}^\flat v^\Lambda)\Big)\,. \tag{A.10}$$

Notice that, in relation to the main text, we have that $2A^\Lambda = \zeta^\Lambda$, as stated before. With this, and using homogeneity of the prepotential F, we compute the second derivatives

$$\mathcal{L}_{\Lambda\Sigma} = -\frac{\mathrm{i}}{2r^\flat}\Big(F_{\Lambda\Sigma} - \bar{F}_{\Lambda\Sigma}\Big), \qquad \mathcal{L}_{\flat\Lambda} = -2A^\Sigma \mathcal{L}_{\Lambda\Sigma}\,, \tag{A.11}$$

and

$$\mathcal{L}_{\flat\flat} = -\frac{\chi_{\mathrm{T}}}{(r^\flat)^2} + 4A^\Lambda A^\Sigma \mathcal{L}_{\Lambda\Sigma}\,. \tag{A.12}$$

These entries are needed to compute the matrix elements of the matrix M. In fact, we need to determine the inverse of M in (A.5). For that, we first need to find the determinant of M, whose general form was given in [61]

$$\det[M] = \frac{1}{3\cdot 4^3}\Big[(\mathcal{L}_{IJ}\vec{r}^I\cdot\vec{r}^J)^3 - \mathrm{Tr}(PQ)\Big], \tag{A.13}$$

where the matrices P and Q are defined as

$$P_{IJ} \equiv \mathcal{L}_{IK}(\vec{r}^K\cdot\vec{r}^L)\mathcal{L}_{LJ}\,, \qquad Q^{IJ} \equiv (\vec{r}^I\cdot\vec{r}^K)\mathcal{L}_{KL}(\vec{r}^L\cdot\vec{r}^J)\,. \tag{A.14}$$

Defining the quantity

$$Y^\Lambda \equiv \frac{v^\flat}{(r^\flat)^{3/2}}\eta_+^\Lambda\,, \tag{A.15}$$

we can compute

$$\begin{aligned}
P_{\Lambda\Sigma} &= \frac{1}{2r^\flat}\Big[(NY)_\Lambda(N\bar{Y})_\Sigma + (NY)_\Sigma(N\bar{Y})_\Lambda\Big],\\
P_{\flat\Lambda} &= -\frac{1}{r^\flat}\Big[(NY)_\Lambda(N\bar{Y})_\Sigma + (NY)_\Sigma(N\bar{Y})_\Lambda\Big]A^\Sigma\,,\\
P_{\flat\flat} &= \frac{\chi_{\mathrm{T}}^2}{(r^\flat)^2} + \frac{4}{r^\flat}(NY)_\Lambda(N\bar{Y})_\Sigma A^\Lambda A^\Sigma\,,
\end{aligned} \tag{A.16}$$

where we used the notation $(NY)_\Lambda = N_{\Lambda\Sigma}Y^\Sigma$. Similarly we find for the matrix elements of Q

$$\begin{aligned}
Q^{\Lambda\Sigma} &= -4(r^\flat)^2\chi_{\mathrm{T}}A^\Lambda A^\Sigma\\
&\quad -2r^\flat[(YN\bar{Y})(Y^\Lambda\bar{Y}^\Sigma + Y^\Sigma\bar{Y}^\Lambda) + Y^\Lambda Y^\Sigma(\bar{Y}N\bar{Y}) + \bar{Y}^\Lambda\bar{Y}^\Sigma(YNY)]\,,\\
Q^{\flat\Lambda} &= -2\chi_{\mathrm{T}}(r^\flat)^2A^\Lambda\,,\\
Q^{\flat\flat} &= -\chi_{\mathrm{T}}(r^\flat)^2\,.
\end{aligned} \tag{A.17}$$

For the first line we used the identity

$$(\vec{r}^{\Lambda} \cdot \vec{r}^{\Sigma}) = 2r^{\flat}(Y^{\Lambda}\bar{Y}^{\Sigma} + Y^{\Sigma}\bar{Y}^{\Lambda}) + 4(r^{\flat})^2 A^{\Lambda}A^{\Sigma}\,, \tag{A.18}$$

which can be proven from the relation

$$\eta_+^{\Lambda}\eta_-^{\Sigma} + \eta_-^{\Lambda}\eta_+^{\Sigma} = \frac{1}{2v^{\flat}\bar{v}^{\flat}}\left(\vec{r}^{\flat} \times \vec{r}^{\Lambda}\right) \cdot \left(\vec{r}^{\flat} \times \vec{r}^{\Sigma}\right). \tag{A.19}$$

Straightforward computation yields

$$\mathrm{Tr}(PQ) = -K^3 - 6|YNY|^2 K\,, \tag{A.20}$$

from which one can find the formula for the determinant (A.21),

$$\det[M] = \frac{1}{32}(YNY)(\bar{Y}N\bar{Y})(YN\bar{Y})\,. \tag{A.21}$$

The inverse matrix M^{-1} was also given in [61]. In our notation, and using (A.14), it can be rewritten as

$$\begin{aligned}(M^{-1})_{rs} &= \frac{32}{\det[M]}\Big[\big(\chi^2 - P_{IJ}(\vec{r}^{I} \cdot \vec{r}^{J})\big)\delta_{rs} + 2(\vec{r}^{I})_r P_{IJ}(\vec{r}^{J})_s\Big] \\ &= -\frac{2}{|YNY|^2 K(Y,\bar{Y})}\Big[\big(K^2 + |YNY|^2\big)\delta_{rs} - (\vec{r}^{I})_r P_{IJ}(\vec{r}^{J})_s\Big]\,.\end{aligned} \tag{A.22}$$

Finally we compute

$$\mathcal{H}_{IJ} = \chi_{\mathrm{T}}\mathcal{L}_{IJ} + \frac{1}{2|YNY|^2}\Big((K^2 + |YNY|^2)P_{IJ} - (PQ)_I{}^K\mathcal{L}_{KJ}\Big), \tag{A.23}$$

and define the dilaton as

$$\mathrm{e}^{\phi} = \frac{K(Y,\bar{Y})}{4r^{\flat}} = -\frac{1}{8(r^{\flat})^3}\mathcal{L}_{\Lambda\Sigma}(\vec{r}^{\flat} \times \vec{r}^{\Lambda}) \cdot (\vec{r}^{\flat} \times \vec{r}^{\Sigma})\,, \tag{A.24}$$

with the same normalization as in [23]. Notice that this coincides with the dilaton given in (3.34) when we identify $\phi = 2U$. We then find for the components

$$\begin{aligned}\mathcal{H}_{\flat\flat} &= -8\mathrm{e}^{\phi}\Big(\mathrm{e}^{\phi} - (\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma}A^{\Lambda}A^{\Sigma}\Big), \\ \mathcal{H}_{\flat\Lambda} &= -4\mathrm{e}^{\phi}A^{\Sigma}(\mathcal{N}+\bar{\mathcal{N}})_{\Sigma\Lambda}\,, \\ \mathcal{H}_{\Lambda\Sigma} &= 2\mathrm{e}^{\phi}(\mathcal{N}+\bar{\mathcal{N}})_{\Lambda\Sigma}\,.\end{aligned} \tag{A.25}$$

This matches precisely with the matrix $\mathcal{T}_{IJ}$ in (A.2) up to an overall normalization factor which is exactly the same as in [23]. One can repeat this analysis for the other terms in the Lagrangian, but these are fixed by supersymmetry, and moreover in [23] they were shown to be correctly reproduced in the SU(2) gauge $v^{\flat} = 0$.

This concludes the proof that the superspace Lagrangian (2.27) describes the c-map Lagrangian (A.1), and validates the "covariant c-map" formulae in section 3.

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